Funding Constraints and Informational Efficiency *

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Abstract

We develop a tractable rational expectations model that allow for general portfolio constraints. We apply our methodology to study a model where constraints arise due to endogenous margin requirements. We argue that margin requirements affect and are affected by informational efficiency, leading to a novel amplification mechanism. A drop in investors' wealth tightens constraints and reduces their incentive to acquire information, which lowers price informativeness. Moreover, financiers who use information in prices to assess the risk of financing a trade face more uncertainty and set tighter margins, which further tightens constraints. This *information spiral* implies that risk premium, conditional volatility and sharpe ratios rise disproportionately as investors' wealth drops. Our model uncovers a new, information-based rationale why the wealth of investors is important.

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1 Introduction

One of the basic tenets of financial economics is that market prices aggregate information of investors. The core of the argument is that investors acquire information about future asset values and trade on it, thereby impounding that information into price. This argument relies on investors' incentives to acquire information and their capacity to trade on it, both of which are crucially affected by their ability to fund their trades. This raises an important question: how do funding constraints faced by investors affect price informativeness? Moreover, since the information in prices can be useful for financiers to assess the risk of financing a trade, another important question is how price informativeness affects funding constraints. To answer these questions, one needs a model in which price informativeness and funding constraints are both jointly determined in equilibrium. In this paper we present and analyze such a model and study its implications for asset prices.

The main challenge to answer these questions is that most of the existing literature on rational-expectation-equilibrium (REE) models, which are instrumental for analyzing informational efficiency, could not accommodate constraints in a tractable manner.[1] The first part of our paper develops a tractable REE model with *general portfolio constraints* that can depend on prices. Our model nests specifications of constraints already analyzed in the literature and opens a broader type of constraints for applications.[2]

Our model is a canonical three-dates REE model which consists of a continuum of investors and a competitive, uninformed market marker to clear the market. The investors trade for profit motives and hedging needs. We assume that investors are constrained to trade some maximal long and short positions and these maximal positions can be any functions of price. Before the trading round occurs, the investors acquire a private signal about the asset payoff. They also know that they will receive a future endowment shock that is correlated

[1]Two important exceptions are Yuan (2005) and Nezafat, Schroder, and Wang (2017), which only analyze a special type of borrowing constraint and short-sale constraint, respectively.

[2]Although we focus on the application of our framework to studying margin constraints, in Appendix B, we apply our framework to the borrowing constraints studied in (Yuan, 2005).

to the risky asset's payoff. If the investors do not face any portfolio constraint, the model is standard: investors' demand is linear in their private signal, the endowment shock, and the price. The equilibrium price itself is linear in aggregate signal and endowment shock. We show that when investors' trading positions are constrained, their desired demand i.e., the amount they would like to trade is still linear. However, their true demand is of the truncated linear form: the desired demand is truncated to the maximal long or short positions. With constraints, the equilibrium becomes *generalized linear*: price is informationally equivalent to a linear combination of aggregate signal and endowment shock. Thus, even though price function is potentially nonlinear, the inference is still tractable.

We then apply our methodology to study how portfolio constraints affect informational efficiency. We show that constraints harm informational efficiency via an *information production channel*.[3] Intuitively, when constraints become tighter, investors can only take smaller positions hence profit less on their private information. In anticipation, they would acquire less information. As all investors ex-ante acquire less information, price becomes less informative about asset fundamentals in equilibrium. We obtain these results using the simple expression we obtain for the marginal value of information of an investor who faces general portfolio constraints.[4] This expression can be useful in a broad class of applications in which investors face portfolio constraints and acquire information.

Next, we study the reverse channel on how informational efficiency affects funding constraints. Motivated by real-world margin constraints as argued in Brunnermeier and Pedersen (2009), we assume investors finance their positions through collateralized borrowing from fi-

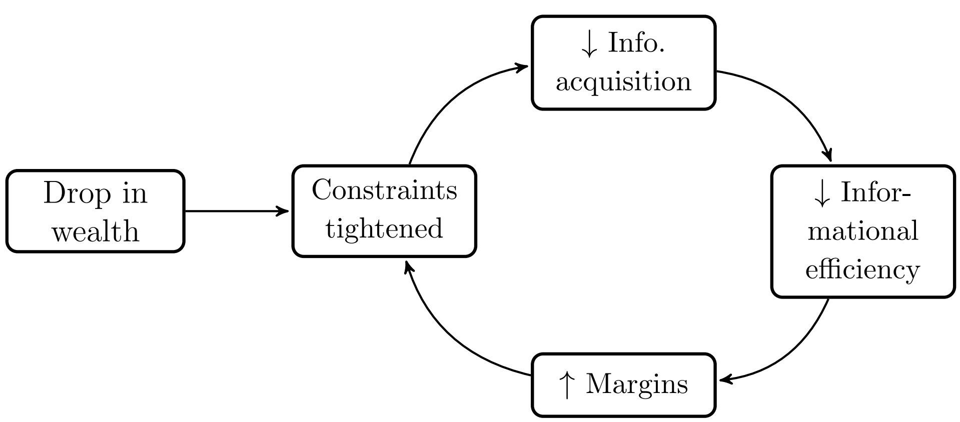
[3]Heuristically, one might expect that the trading constraints per se could reduce informational efficiency because demands of constrained informed investors cannot respond to their private information. We show that this heuristic is not correct because even in the constrained setting, the aggregate demand from investors continues to vary with fundamentals via the changes in the fraction of investors being constrained in long and short positions. When the private information about the asset are more favorable, there will be more buyers and fewer sellers. As a result, the aggregate demand reveals the same amount of information as in the unconstrained setting.

[4]The expression says that the ratio of marginal values of information for a constrained and unconstrained investor is equal to the ratio of utility a constrained investor gets in the states when his constraints do not bind to his total expected utility.

nanciers who require the margins to control their value-at-risk(VaR). We further assume that financiers use information in prices when setting the margin requirement. We argue that lower informational efficiency leads to tighter margins. The intuition is that, when prices are less informative, financiers who use information in prices to assess the risk of financing a trade face more uncertainty about fundamentals and thus set higher margins.

Our model implies that funding constraints affect and are affected by informational efficiency. In light of this, both margins and asset prices are determined jointly in equilibrium: investors and financiers determine demands and margins anticipating a particular price function and, in equilibrium, demands and margins are consistent with the anticipated one. We get our main result, a novel *information spiral* showed in Figure 1. With a negative wealth shock, constraints tighten, investors acquire less information, leading to lower informational efficiency in equilibrium. As price becomes less informative of the fundamentals, financiers tighten their margins requirement to satisfy their VaR constraints, further tightening investors' funding constraints. As a result, a small shock to wealth may have a profound effect on information production, informational efficiency and funding constraints.

Figure 1: Amplification loop



Our information spiral suggests a novel amplifying mechanism on asset prices. We show that a small shock to investors' wealth can lead to large increase in conditional volatility, risk premium and sharpe ratio of the asset. Each of these results match empirical observations during crises.[5] While the literature has proposed other amplifying mechanisms for the effect of wealth shocks, ours is unique in the sense that it acts through informativeness of the financial markets, which could have further macro-economic consequences given the central role of the stock market in the real economy.

1.1 Related Literature

This paper lies at the intersection of various strands of literature. On the one hand, we share the emphasis of the work that studies the role played by financial markets in aggregating and disseminating information, following Grossman (1976), Grossman and Stiglitz (1980), Hellwig (1980a) and Diamond and Verrecchia (1981). In most of these models, it is generally assumed that investors can borrow freely at the riskless rate i.e., no funding constraints. From the methodological perspective, we contribute to this literature by developing a REE model that can incorporate general portfolio constraints. Similar to us, Yuan (2005) studies REE model with linear price dependent constraints. Our model also nests the model of Nezafat et al. (2017) which studies how short sale constraints affect information acquisition and asset prices.

Our work is related to the literature on information acquisition in REE models. Peng and Xiong (2006); Van Nieuwerburgh and Veldkamp (2009); Van Nieuwerburgh and Veldkamp (2010) study financial investor's information acquisition problem without funding constraints. On the contrary, we study information acquisition incentives with funding constraints. We show that funding constraints affects and are affected by informational efficiency (through information acquisition of investors) which leads to an emergence of information spiral. Our paper also relates to the recent literature on the role of secondary financial markets as a primary source of information for decision makers. See Bond, Edmans, and Goldstein (2012) for recent survey on this topic. Goldstein, Ozdenoren, and Yuan (2013) show that the feedback effect from

[5] Financial crises, such as the hdge fund crisis of 1998 or 2007/2008 subprime crisis, have several common characteristics: risk premia rise, conditional volatility of asset prices rise and sharpe ratio rises.

asset prices to the real value of a firm because capital providers learn from prices, generating complementarities in investments. Dow, Goldstein, and Guembel (2017) show that the feedback effect generates complementarities in the decision to produce information, but not in the trading decision. We contribute to this literature to study how financiers of investors can use the information in prices to set the margins requirement and we find that lower informational efficiency leads to tighter margins.

Finally, our work contributes to the literature on the effect of investors' wealth and the associated amplification mechanisms. For example, Xiong (2001) studies wealth constraint as an amplification mechanism, while Kyle and Xiong (2001) study it as a spillover mechanism. Gromb and Vayanos (2002, 2017) develop an equilibrium model of arbitrage trading with margin constraints to explain contagion. Brunnermeier and Pedersen (2009) studies how funding liquidity and market liquidity reinforce each other. He and Krishnamurthy (2011), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014) study how a fall in intermediary capital reduces their risk-bearing capacity and lead to rises in risk premium and conditional volatility. Overall, this literature emphasizes that leverage and asset prices need to be jointly determined (see review article by Fostel and Geanakoplos (2014)). Our paper is complementary to these studies. We show that informational efficiency amplifies the wealth effect n a REE model with a model of endogenous margin requirements. Our mechanism is novel in the sense that it involves changes in stock-market informativeness, which should be an important channel given the central role of the stock market in the real economy.

The reminder of the article is organized as follows. In section 2, we solve for the financial market equilibrium and value of information in an REE model with general portfolio constraints. In section 3, we introduce margin requirements and argue that funding constraints affect informational efficiency. In section 4, we endogenize the margin constraints and argue that informational efficiency affects funding constraints. Because of this, information spiral emerges. In section 5, we explore the implications of this spiral for asset prices. Section 6 concludes.

2 An REE model with general portfolio constraints

In this section we introduce and solve the model with exogenous portfolio constraints. In section 4 we endogenize the constraints and solve for a full equilibrium of the model.

2.1 Setup

There are three dates (i.e., $t \in \{0, 1, 2\}$) and two assets. The risk-free asset is the numeriaire. The payoff (fundamental value) of the risky asset is $v \sim N(\bar{v}, \tau_v^{-1})$ which is paid at date 2, and the aggregate supply of the asset is assumed to be constant 1 unit. The economy is populated by a unit continuum of investors, indexed by $i \in [0, 1]$, with identical CARA preferences over terminal wealth with risk aversion γ. Investors acquire information at date 0, trade the risky asset at $t = 1$, and consume their assets' payoffs at $t = 2$.

At date 2, each investor receives an endowment $e_i v$, where e_i is privately known to investor i at the trading date ($t = 1$).[6] We assume that endowment shocks e_i have aggregate and idiosyncratic components, $e_i = z + u_i$, where u_i are iid, with $u_i \sim N(0, \tau_u^{-1})$, $z \sim N(0, \tau_z^{-1})$ and both z and u_i are independent of all the other random variables in the model. Differences in exposures "e_i" across investors motivates trade in risky asset.

At date 1, each investor i receives a signal $s_i = v + \epsilon_i$, where ϵ_i are iid with $\epsilon_i \sim N(0, \tau_{\epsilon_i}^{-1})$. which he decides to acquire at date 0. The precision of her private signal τ_{ϵ_i} is optimally chosen by investor i at date 0, subject to a cost function $C(.)$. We assume that the cost function is identical to all investors and possesses standard characteristics: C is continuous, $C(0) = C'(0) = 0$, and $C', C'' > 0$ for all τ_{ϵ_i}. The information set of investor i at time 1 is $\mathcal{F}_i = \{s_i, e_i, p\}$, where p is the equilibrium price at time 1. There is also a competitive market maker with quadratic inventory holding costs.[7] The market maker has neither endowment

[6] When we introduce wealth effects later, we also assume that these endowment shocks are not pledgeable at $t = 1$ and hence not part of their wealth.

[7] Our results hold with a CARA market maker. We resort to current specification of market maker preferences for expositional convenience.

shocks nor private information about the asset payoff. He takes prices as given, learns from it and submits his demands for the risky asset.

Constraints: Investors, but not the market maker are subject to the following funding constraints: given the price p, the minimum and maximum positions that an investor can take are, respectively, $a(p)$ and $b(p)$. The functions $a(p)$ and $b(p)$ may depend on investors' initial wealth W_0 and the aggregate equilibrium parameters, such as price volatility. Where it does not cause confusion, we will not indicate this dependence explicitly. To summarize, at date 1 investors solve the following problem

$$\max_{x_i(p;s_i,e_i)\in[a(p),b(p)]} E[-\exp(-\gamma W_i)|s_i, e_i, p],$$

$$\text{subject to: } W_i = W_0 + x_i(v-p) + e_i v,$$

The market maker solves

$$\max_{x_m(p)} x_m(p)E(v-p|p) - \frac{\kappa_m}{2}(x_m(p))^2.$$

where $\kappa_m \geq 0$ is his marginal inventory cost. Finally, the equilibrium price is set to clear the market:

$$\int x_i(p;s_i,e_i)di + x_m(p) = 1$$

2.2 Financial Market Equilibrium

We first solve for equilibrium in the unconstrained setting, which has already been studied in Ganguli and Yang (2009) and Manzano and Vives (2011). This is an important step in characterizing the equilibrium with constraints, which is why we review it here.

2.2.1 Unconstrained setting

We summarize the most important results about the unconstrained setting in the proposition below.

Proposition 1. *(Unconstrained equilibrium) Suppose* $\gamma^2 > 4\tau_\epsilon\tau_u$. *Then there exists a unique stable equilibrium in which the price is informationally equivalent to a sufficient statistic* $\phi^u = v - \frac{z}{\beta^u}$, *which can be computed from price as follows:* $\phi^u = f_0^u + f_1^u p$. *The aggregate demand of investors and market maker can be written as*

$$X^u = c_0 + c_\phi \phi - c_p p, \qquad x_m = c_0^m + c_\phi^m \phi - c_p^m p.$$

The individual demand of investor i *can be written as follows*

$$x_i^u = X^u + \xi_i, \ \xi_i \sim N(0, \sigma_\xi^2),$$

where

$$\xi_i = \frac{\tau_\epsilon \epsilon_i + (\beta\tau_u - \gamma)u_i}{\gamma}, \ \sigma_\xi^2 = \frac{\tau_\epsilon + (\beta\tau_u - \gamma)^2/\tau_u}{\gamma^2}.$$

The equilibrium informational efficiency is given by

$$\beta = \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{\tau_\epsilon}{\tau_u}}. \tag{1}$$

The coefficients are reported in the appendix.

Note that β is a key equilibrium variable that we will focus on. As we discuss below, β reflects how much information about the dividend of the risk asset v is contained in prices. It captures the informational efficiency of the market. Importantly, from equation 1, we can see that informational efficiency increases with τ_ϵ, the precision of signals of investors.

Lemma 1. *The unconditional risk premia is given by*

$$E(v-p) = \frac{1}{\frac{\tau}{\gamma} + \frac{1}{\kappa_m}} \tag{2}$$

where $\tau = \tau_v + \tau_\epsilon + \beta^2(\tau_z + \tau_u)$.

Informed trading affects the risk premia by changing the amount of information revealed in prices and affecting posterior uncertainty of investors. Informational efficiency β decreases the risk premium in the economy (via τ) because investors face less risk by learning from more informative prices.

2.2.2 Constrained setting

We now impose the constraints $a(p)$ and $b(p)$ into investors problem. We guess and later verify, that there exists a *generalized linear equilibrium* in the economy, which we define as follows.

Definition 1. *An equilibrium is called generalized linear if there exists a function* $f(p)$ *and a constant* β*, such that* $\phi = v - \frac{z}{\beta}$ *is a sufficient statistic for price and is given by* $\phi = f(p)$.

In an equilibrium as defined above, despite the potential nonlinearity of the price function, the sufficient statistic ϕ is still normally distributed. Thus, the inference from price is tractable. Given normality, the conditional distribution of v given p is fully characterized by two moments, mean and variance. In particular,

$$Var(v|p) = (\tau_v + \beta^2\tau_z)^{-1}.$$

The conditional variance decreases in the signal-to-noise ratio β. For this reason we call β *informational efficiency.* We characterize the conditional mean $E[v|p]$ function by its sensitivity to changes in prices. The latter can be calculated as follows

$$\frac{dE[v|p]}{dp} = \frac{\beta^2\tau_z}{\tau_v + \beta^2\tau_z} f'(p).$$

Thus, up to a constant $\frac{\beta^2\tau_z}{\tau_v+\beta^2\tau_z}$ (which is increasing in informational efficiency β) the sensitivity of the conditional mean $E[v|p]$ is given by $f'(p)$. Therefore, we call the function $f'(p)$ *information sensitivity*. Intuitively, it shows how large a change of the sufficient statistic ϕ corresponds to a unit change in prices. As we will show, constraints do affect this sensitivity, which in turns affect risk premium and volatility of returns.

We first show below that, there exists a generalized linear equilibrium in an economy with constraints. Moreover, the informational efficiency in the constrained economy is equal to that in the unconstrained economy, i.e. with $\beta = \beta^u$. To proceed we define a function $T(x; a, b)$ that truncates its' argument x to the interval $[a, b]$:

$$T(x; a, b) = \begin{cases} x, & \text{if } a < x < b, \\ b, & \text{if } x > b, \\ a, & \text{if } x < a. \end{cases}$$

The demand of an investor i in the conjectured equilibrium can be written as

$$\begin{aligned} x_i &= T\left(x_i^u; a(p), b(p)\right) \\ &= T\left(X^u + \xi_i; a(p), b(p)\right). \end{aligned}$$

where the x_i^u and X^u denotes demand of investor i and aggregate demand in the unconstrained economy.

The market clearing condition now can be written as

$$\int x_i di + x_m(p) = 1 \implies \int T\left(X^u + \xi_i; a(p), b(p)\right) di + x_m(p) = 1.$$

Even though the market clearing condition looks intimidating, one can find a unique generalized linear equilibrium in this setup. The following proposition characterizes the equilibrium.

Proposition 2. *(Equilibrium with portfolio constraints) Suppose investors face position constraints and have identical signal variances τ_ϵ^{-1}. Then there is unique stable equilibrium with informational efficiency $\beta = \beta^u$. In particular, there is a unique function $f(\cdot)$ such that*

$$f(p) = \phi \equiv v - (\beta^u)^{-1} z.$$

The function $f(p)$ satisfies the ODE

$$f'(p) = \frac{c_p^m + \pi_2 c_p - \pi_1 a'(p) - \pi_3 b'(p)}{c_\phi^m + \pi_2 c_\phi} \tag{3}$$

subject to the the boundary condition $f(0) = f_0$, where $\pi_1(\phi, p) = \Phi\left(\frac{a(p) - X^u(p,\phi)}{\sigma_\xi}\right)$ is the fraction of investors for whom the lower constraint binds, $\pi_3(\phi, p) = 1 - \Phi\left(\frac{b(p) - X^u(p,\phi)}{\sigma_\xi}\right)$ is the fraction of investors for whom the upper constraint binds, and $\pi_2 = 1 - \pi_1 - \pi_3$ is the fraction of unconstrained investors. The constant f_0 is the unique solution to

$$g(f_0, 0) + c_0^m + c_\phi^m f_0 = 1.$$

The aggregate demand of investors is given by

$$g(\phi, p) = \pi_1 a(p) + \pi_3 b(p) + \pi_2 X^u + \sigma_\xi \left(\Phi'\left(\frac{a(p) - X^u}{\sigma_\xi}\right) - \Phi'\left(\frac{b(p) - X^u}{\sigma_\xi}\right)\right).$$

Proof. We derive two results here: (1) that the price is informationally equivalent to $\phi = v - \frac{z}{\beta^u}$

(2) the expression for information sensitivity $f'(p)$. The rest of the results are derived in the Appendix.

By the exact law of large numbers one can write the aggregate demand of investors as

$$X = \int x_i di = E_{\xi_i} \left[T \left(X^u + \xi_i; a(p), b(p) \right) \right].$$

For a given price p, X is an increasing (and thus invertible) function of X^u. Therefore given p one can compute $X = 1 - x_m(p)$, from which one can infer X^u, from which, in turn, one can express $\phi^u = v - \frac{z}{\beta^u}$. This proves the first result.

It remains to find an expression for the information sensitivity function $f'(p)$. Differentiating market-clearing condition implicitly one can get

$$f'(p) = \frac{d\phi}{dp} = -\frac{\frac{\partial}{\partial p}(\text{Aggregate Demand})}{\frac{\partial}{\partial \phi}(\text{Aggregate Demand})}.$$

For the numerator we have

$$\frac{\partial}{\partial p}(\text{Aggregate Demand}) = c_p^m + \pi_2 c_p - \pi_1 a'(p) - \pi_3 b'(p).$$

The sensitivity of aggregate demand with respect to p comes from four sources. First, there is a market maker, who contributes c_p^m. Second, there is a fraction π_2 of unconstrained investors, each contributing c_p. Third there is a fraction π_1 of investors whose lower constraint $a(p)$ binds. Each of them contributes $a'(p)$ to aggregate price sensitivity. Finally, there is a measure π_3 of investors whose upper constraint $b(p)$ binds. Each of them contributes $b'(p)$ to aggregate price sensitivity.

By a similar argument, for the denominator, we have

$$\frac{\partial}{\partial \phi}(\text{Aggregate Demand}) = c_\phi^m + \pi_2 c_\phi.$$

It remains to determine fractions π_1, π_2 and π_3. For π_1 we can write

$$\pi_1(\phi, p) = Pr\left(X^u(p, \phi) + \xi_i < a(p)\right) = \Phi\left(\frac{a(p) - X^u(p, \phi)}{\sigma_\xi}\right),$$

where $\Phi(\cdot)$ denotes the CDF of a standard normal random variable. The expressions for π_2 and π_3 can be derived analogously. ■

Observation 1: *Constraints do not affect price informativeness with exogenously specified information.* A recent paper Nezafat et al. (2017) has demonstrated this result for a special case of short-sale constraints: $a(p) = 0$, $b(p) = \infty$. The proof above highlights the intuition for this result: the *aggregate* constrained and unconstrained demands of investors are informationally equivalent. Since the informational content of the price is solely determined by trading of investors (because only they have private information), price reveals the same amount of information as in the unconstrained setting. Later on, when we endogenize information acquisition we will see that constraints affect price informativeness by changing incentives of investors to acquire information. However even with exogenously specified information, the above result does not imply that constraints do not affect the inference from prices at all.

Observation 2: *Constraints do affect information sensitivity $f'(p)$.* It will be easier to illustrate the effects assuming constraints do not depend on prices, $a(p) = a$, $b(p) = b$. In that case we have

$$f'(p) = \frac{c_p^m + \pi_2 c_p}{c_\phi^m + \pi_2 c_\phi}.$$

We see that constraints affect the fraction π_2 of unconstrained investors. In the case when constraints do not bind we get $f'(p) = \frac{c_p^m + c_p}{c_\phi^m + c_\phi}$. When constraints almost always bind we get $f'(p) = \frac{c_p^m}{c_\phi^m}$. When constraints are tighter, π_2 decreases and information sensitivity moves towards $\frac{c_p^m}{c_\phi^m}$. When constraints are looser π_2 increases and, information sensitivity moves towards $\frac{c_p^m + c_p}{c_\phi^m + c_\phi}$.

Observation 3: *Linear equilibrium in special cases.* (a) When market-maker is risk-neutral and has no inventory cost ($\kappa_m = 0$) we have the following corollary.

Corollary 1. *If the market maker's inventory cost is zero, price function is linear and independent of the constraints:*

$$p = \frac{\tau_v \bar{v} + \beta^2 \tau_z \phi}{\tau_v + \beta^2 \tau_z} \tag{4}$$

When market maker has no inventory cost, he sets the price $p = E[v|p]$. Moreover, Proposition 2 implies that information in price remains the same with or without constraints and hence price can be written as $p = E[v|\phi]$, which yields equation 4.

(b) When the constraints do not depend on prices, $a(p) = a$, $b(p) = b$ and there is no market maker (or $\kappa_m^{-1} = 0$, i.e inventory cost is infinite) we have the following corollary.

Corollary 2. *If $\kappa_m^{-1} = 0$, and constraints do not depend on p, $a'(p) = b'(p) = 0$ equilibrium is linear*

$$f'(p) = \frac{c_p}{c_\phi},$$

Moreover, neither price informativeness β, nor information sensitivity $f'(p)$ are affected by constraints.

These special cases could be useful for the study of short-sale constraint as in Nezafat et al. (2017) (with $a = 0$ and $b = \infty$) and constant dollar margins m^+ and m^- (with $a = -\frac{W_0}{m^-}$ $b = \frac{W_0}{m^+}$).

2.3 Value of information

Up till now, we take the investors' signals as given and solved for the financial market equilibrium at $t = 1$. In this section, we study the incentives of investors to acquire information at $t = 0$. We will derive the expression for marginal value of information with general portfolio constraints and demonstrate that if constraints are tightened only for one investor, her marginal value of information decreases.

At date 0, each investor chooses the precision of her private signal τ_{ϵ_i} to maximize her expected utility from trade:

$$E[-\exp(-\gamma(W_0 + x_i(v-p) + e_i v - C(\tau_{\epsilon_i})))] \quad \text{where} \quad x_i = T(x_i^u, a(p), b(p))$$

where $C(\cdot)$ is the cost of acquiring information. The certainty equivalent at time 1 can be written as

$$CE_1 = W_0 + (x_i + e_i)\left(E[v|\mathcal{F}_i] - p\right) - \frac{\gamma}{2\tau_i}(e_i + x_i)^2 + e_i p - C(\tau_{\epsilon_i}).$$

where $\tau_i = \tau_{\epsilon_i} + \tau_v + \beta^2(\tau_z + \tau_u)$. Next, we note that

$$x_i^u + e_i = \frac{\tau_i}{\gamma}(E[v|\mathcal{F}_i] - p) \Rightarrow E[v|\mathcal{F}_i] - p = \frac{\gamma}{\tau}\left(x_i^u + e_i\right)$$

where x_i^u is her demand in the unconstrained economy. Substituting this into the certainty equivalent, we get

$$CE_1 = \underbrace{-\frac{\gamma}{2\tau}\left(x_i^u - x_i\right)^2}_{\text{the term due to constraints}} + W_0 + \frac{\gamma}{2\tau}\left(x_i^u + e_i\right)^2 + e_i p - C(\tau_{\epsilon_i})$$

We see that the time-1 change in the certainty equivalent due to introduction of constraints is captured by a single term, $-\frac{\gamma}{2\tau}\left(x_i^u - x_i\right)^2$, which captures the “distance” between the demand with and without constraints. We note that CE_1 decreases as constraints become tighter.

Define time-0 certainty equivalent as the solution to $e^{-\gamma CE_0} = E[e^{-\gamma CE_1}]$. We define the *marginal value of information* as $\frac{dCE_0}{d\tau_{\epsilon_i}}$. The next proposition characterizes the marginal value of information with general constraints.

Proposition 3. *(Marginal value of information) The marginal value of information is given* *by*

$$\text{MVI} = \frac{1}{2\tau_i} \underbrace{\frac{U_0^u}{U_0}}_{\text{the term due to constraints}}, \tag{5}$$

where $\tau_i = \tau_{\epsilon_i} + \tau_v + \beta^2(\tau_u + \tau_z)$ *is the total precision of information available to investor* i, $U_0^u = E[-e^{-\gamma CE_1}\mathbb{I}(x_i^u = x_i)]$ *is the expectation of utility in the states when constraints do not bind and* $U_0^u = E[-e^{-\gamma CE_1}]$ *is time-0 expected utility.*

We see that the contribution of constraints to marginal value of information is captured by the term $\frac{U_0^u}{U_0} \in [0, 1]$: when there are no constraints this term is equal to 1, when constraints almost always bind, this term is close to zero. Because with constraints $\frac{U_0^u}{U_0} < 1$ we have the following corollary.[8]

Corollary 3. *The marginal value of information with constraints is smaller than that without constraints.*

The above corollary implies that with constrained investors have less incentives to acquire information (compared to the case of no constraints at all). Next, we demonstrate that as *constraints become tighter* for just one investor (so that price distribution remains unchanged), her marginal value of information decreases.

Proposition 4. *Suppose that all investors face portfolio constraints* $a(p)$ *and* $b(p)$*. Suppose that for an investor* i*, constraints become tighter, i.e.* $a(p)$ *and* $b(p)$ *becomes* $\hat{a}(p)$ *and* $\hat{b}(p)$ *such that* $\forall p$, $\hat{b}(p) \leq b(p)$ *and* $\hat{a}(p) \geq a(p)$*. Then the marginal value of information of investor* i *decreases. Moreover, if* $\kappa_m = 0$ *and if all investors constraints become tighter (from* $[a(p), b(p)]$ *to* $[\hat{a}(p), \hat{b}(p)]$*), the marginal value of information for each of them decreases.*

The above proposition illustrates one of the key forces of our mechanism: with tighter constraints, investors have less incentives to acquire information. In the first part of the above proposition, the constraints change for just one investor. This greatly simplified our proof

[8] Nezafat et al. (2017) demonstrate this result for the case of short-sale constraints.

as the prices remained unchanged (as in the case of $\kappa_m = 0$). When constraints change for all investors and market-maker has positive inventory costs, prices change and such general proposition cannot be proven anymore.

3 Portfolio constraints from margin requirements

So far we have studied general portfolio constraints. In this section, we put more structure and assume that constraints arise from margin requirements. In particular, we assume that to open a long position, an investor has to put $m^+ \geq 0$ of his capital (wealth), per unit of asset. Similarly, to establish a short position an investor has to put m^-. Investors face the constraint that the total margin on their position cannot exceed their initial wealth:

$$m^-[x_i]^- + m^+[x_i]^+ \leq W_0,$$

where $[x_i]^-$ and $[x_i]^+$ are the positive and negative parts of x_i respectively. This implies that the constraints on positions can be written as:

$$a(p) = -\frac{W_0}{m^-}, \quad b(p) = \frac{W_0}{m^+}. \tag{6}$$

For now, we assume that m^+ and m^- are constants and do not depend on price. We will relax this assumption when we endogenize margins in the next section. Here, we analyze how funding constraints from margin requirements affect informational efficiency, through the investors' information acquisition decision in equilibrium.

Proposition 5. *(Equilibrium information acquisition under margin requirements) Given margins m^+ and m^-, the equilibrium precision τ_e^* satisfies the following equation:*

$$\gamma C'(\tau_e^*) = \underbrace{\frac{1}{2\tau}}_{GS\ term} \underbrace{\frac{U_0^u(\tau_e^*, \tau_e^*, W_0)}{U_0(\tau_e^*, \tau_e^*, W_0)}}_{Effect\ of\ constraints} \tag{7}$$

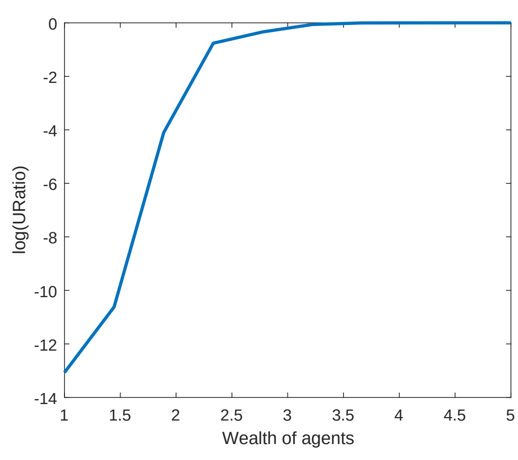
where $\tau = \tau_e^* + \tau_v + \beta^2(\tau_u + \tau_z)$ is the equilibrium posterior precision and the functions $U_0^u()$ and $U_0()$ are defined in the appendix.

When $\kappa_m = 0$, the right hand side of equation 7 decreases with wealth for every τ_e.

The right hand side of equation 7 characterizes the equilibrium marginal benefit of acquiring information. Note that it has two terms: the first term is the standard term present in Grossman and Stiglitz (1980), which says that marginal benefit increases as posterior uncertainty increases. The second term is the effect of constraints. This term is 1 when investors are unconstrained and 0 when investors are fully constrained. When the market maker has no inventory cost, the proposition shows that the ratio decreases with wealth. When the market maker has positive inventory cost, the effect is too complicated to study analytically. We proceed numerically to analyze this case. Figure 2 plots how this ratio changes with wealth of investors. Note that as investors become more constrained, this ratio decreases which decreases the marginal benefit of acquiring information.

Figure 2: Value plot

The figure plots the log(ratio of utilities) shown in equation 7. Other parameter values are set to: $\tau_u = \tau_Z = \tau_v = \tau_e = 1$, $\bar{v} = 4$,$\gamma = 3$,$\kappa_m = 1.5$.



Proposition 5 also suggests that wealth effect arise in our model with constraints via information acquisition. As wealth of investors decrease, they become more constrained and acquire less information. This reduces informational efficiency of prices. It is well known that there are no wealth effects in standard CARA-normal models. To the best of our knowledge, ours is the first noisy REE model that admits closed form solutions with wealth effects and margin constraints.

4 Equilibrium with endogenous margin constraints

Up until now we assumed that margins are exogenous. In this section, we endogenize margins as in Brunnermeier and Pedersen (2009). We assume the financiers use information from prices to set margin in order to control Value-at-Risk (henceforth VaR):

$$m^{+}(p) = \inf\{m^{+}(p) \geq 0 : Pr(p - v > m^{+}(p)|p) \leq 1 - \alpha\}.$$

$$m^{-}(p) = \inf\{m^{-}(p) \geq 0 : Pr(v - p > m^{-}(p)|p) \leq 1 - \alpha\}.$$

$m^{+}(p)$ and $m^{-}(p)$ are the margins on long and short positions (per unit of asset) respectively. Intuitively, the financiers require the investors to set aside a minimum amount of cash, i.e. margin, which is just large enough to sufficiently cover the potential loss from trading with probability α. We assume that financier is uninformed but can set margin condition on prices. As detailed in the Appendix A of Brunnermeier and Pedersen (2009), this margin specification is motivated by the real-world margin constraints faced by hedge funds and capital requirements imposed on commercial banks.

4.1 Financial market equilibrium with endogenous margin constraints

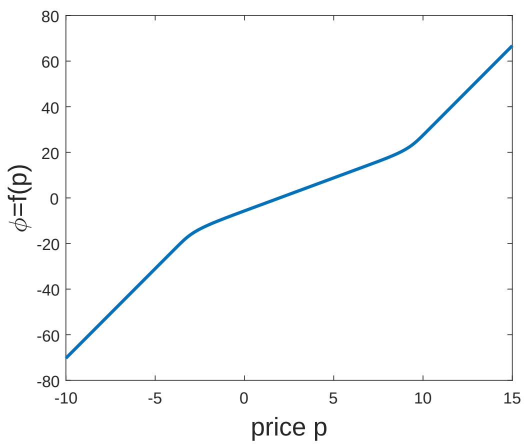
Formally, our financial market equilibrium with endogenous margin constraints is defined as follows: (1) financiers and investors determine demands and margins anticipating a particular price function (2) in equilibrium demands and margins are consistent with anticipated price function. We hold the precisions of investors' signals fixed.

Proposition 6. *(Equilibrium with endogenous margin requirements) When the portfolio constraints are of the form of margin as in equation (6) and margins are endogenously determined by VaR, there exists a unique generalized linear equilibrium. Moreover, in this unique equilibrium the function $f(p)$, i.e. the sufficient statistic ϕ, is increasing in price.*

Figure 3 plots the equilibrium informational content of price ϕ as a function of price. Note that even though it is non-linear, the function is monotonic and hence invertible.

Figure 3: Price plot

The figure plots information in price as a function of price. Other parameter values are set to: $\tau_u = \tau_Z = \tau_v = \tau_e = 1$, $\bar{v} = 4$,$\gamma = 3$,$\kappa_m = 1.5$.



In the previous section, we show that exogenous funding (margin) constraints affect informational efficiency. Next, we examine how informational efficiency can affect margin constraints.

4.2 Informational efficiency affects tightness of funding constraints

We now derive the expression for margins. To compute $m^+(p)$, we make use of the definition of conditional risk premium $rp(p)$ that $p = E[v|p] - rp(p)$ and first determine the functions $m_n^+(p)$

that satisfies

$$
\begin{aligned}
1-\alpha &= Pr(E[v|p] - rp(p) - v > m_n^+(p)|p) \\
&= Pr\left(\sqrt{\tau_m}(E[v|p] - v) > \sqrt{\tau_m}(m_n^+(p) + rp(p))\big|p\right) \\
&= 1 - \Phi\left(\sqrt{\tau_m}(m_n^+(p) + rp(p))\right).
\end{aligned}
$$

Thus, we find

$$
m^+(p) = [m_n^+(p)]^+ = \left[\frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}} - rp(p)\right]^+ \tag{8}
$$

Similarly, one can define $m_n^-(p)$ which satisfies $Pr(v - p > m_n^-(p)|p) = 1 - \alpha$ and get

$$
m^-(p) = [m_n^-(p)]^+ = \left[\frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}} + rp(p)\right]^+ \tag{9}
$$

The endogenous VaR margins are determined by three variables. Both margins on long and short positions increase in the exogenous level of confidence α and decrease in the endogenous informational efficiency of price β (through $\tau_m = \tau_v + \beta^2\tau_z$). In addition, the margin on long (short) position decreases (increases) in the endogenous risk premium $rp(p)$. We would like to emphasize the fact that informational efficiency of price affects the tightness of margin constraint.

In equilibrium, conditional risk premium is given by

$$
rp(p) = \kappa_m x_m = \kappa_m(c_0^m + c_\phi^m f(p) - c_p^m p). \tag{10}
$$

Intuitively, the risk premium in the economy is proportional to the demand absorbed by the uninformed market maker (x_m) and is higher when the market maker is more risk averse (higher κ_m). Next, we show that changes in price informativeness affect margins:

Proposition 7. *(Informational efficiency affects constraints) Suppose $\kappa_m = 0$. For a given pair of aggregate state variables (v, z) and a given investors' wealth W_0, when informational* *efficiency (β) decreases, margins (both m^+ and m^-) increase. This implies that the lower constraint (a) increases and the upper constraint (b) decreases. In other words, as informational efficiency drops, constraints become tighter.*

The intuition is as follows. Financiers use information in price to help to asses the risk that the loss from financing exceeds the margin. With less informative prices, financiers face more uncertainty about fundamentals and thus perceive higher risk of financing a trade and require higher margins (direct effect). However, changing informational efficiency also leads to change in risk premium (indirect effect), and this effect decreases m^+ and increases m^-. However, this effect is shut down by the assumption of $\kappa_m = 0$. In our numerical analysis we find that this result still holds even with $\kappa_m > 0$.

4.3 Amplification

In section 3, we did a partial equilibrium analysis and argued that given margins, tighter funding constraints (decrease in wealth) leads to lower informational inefficiency and in section 4, we argued that, given wealth level, lower informational inefficiency leads to tighter margins. In this section, we put all the links together.

First, we show that, with exogenous information, drop in wealth leads to tighter constraints. Note that the constraints are given by

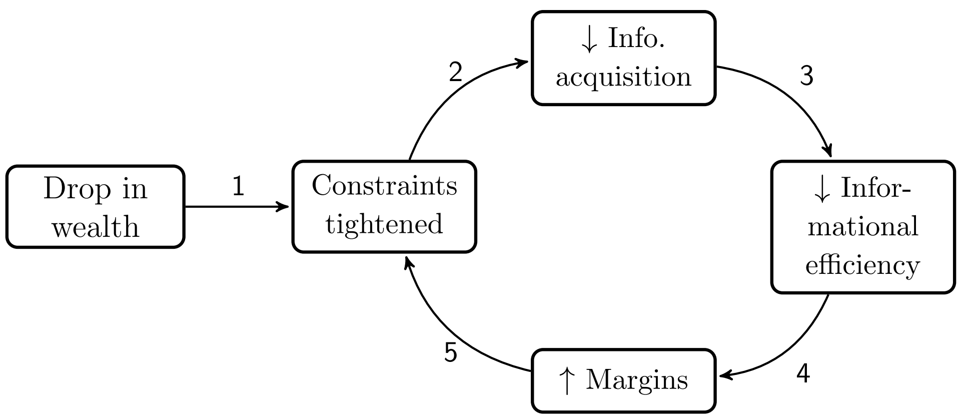
$$a(p) = \frac{W_0}{m^+(p)} \quad b(p) = \frac{W_0}{m^-(p)}$$

Even with exogenous information, as wealth drops, prices change and both numerator(direct effect) and denominator (indirect effect) of the constraints change. But we argue that direct effect always dominate and constraint tightens as wealth drops.

Lemma 2. *(Aggregate investors' wealth relaxes constraints on demand in equilibrium). Suppose that initial wealth W_0 drops for all investors. Then, in any given aggregate state (v, z), a* *increases and b decreases.*

The lemma above establishes the first link in the Figure 4. The second and third links follow from Proposition 5. The fourth and fifth links follow from Proposition 7. Putting all the links together, we will have a amplification loop. We call this *information spiral*. The main implication of the information spiral is that small changes in underlying funding conditions can lead to sharp reductions in information production and hence, informational efficiency. Next, we discuss the implications of this feedback loop for asset prices i.e., risk premium, conditional volatility and Sharpe ratio.

Figure 4: Amplification loop



5 Asset pricing implications

In this section, we will derive the implications of drop in wealth on the equilibrium risk premium and volatility of risky assets. The main result is that a drop in wealth leads to large increase in risk premium, volatility and Sharpe ratio.

5.1 Risk premium

Note that conditional risk premium is given by

$$rp(p) = \kappa_m x_m = \kappa_m(c_0^m + c_\phi^m f(p) - c_p^m p).$$

Hence the unconditional risk premium is given by

$$E[v - p] = E[rp(p)] = \kappa_m(c_0^m + c_\phi^m \bar{v} - c_p^m E(p)).$$

Note that price is a non-linear function of fundamentals and hence we proceed with numerical analysis.

Figure 5: Risk premium

The figure plots risk premium as a function of precision of investors signal for different levels of wealth: $w = 0.2$ and $w = 2$. Other parameter values are set to: $\tau_u = \tau_Z = \tau_v = \tau_e = 1$, $\bar{v} = 4$,$\gamma = 3$,$\kappa_m = 1.5$.

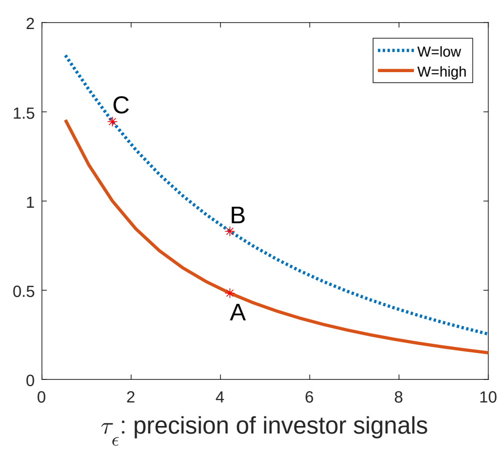


Figure 5 plots the unconditional risk premium in our model against τ_ϵ, precision of investors' signals for two different levels of wealth. Suppose that the point A corresponds to the equilibrium with high wealth level. At this wealth, most of the investors are unconstrained and the equilibrium is close to the unconstrained setting. Now imagine lowering wealth. With lower wealth investors' capacity to long or short asset is diminished due to tighter constraints, which has a similar effect to lowering their risk-bearing capacity (increasing risk aversion). Therefore, the risk premium rises. This argument implies that absent the information production channel (i.e., holding τ_ϵ fixed), the wealth drop would cause an increase in risk premium that corresponds to the move from solid line (corresponding to high wealth level) to dashed line (corresponding to low wealth level), from point A to point B. Moreover, because of information spiral, investors in equilibrium acquire less information, which leads to an additional increase in risk premium, corresponding to the move from point B to point C along the dashed line. Thus, an effect of drop in wealth on risk premium is amplified through the information production channel and the equilibrium moves from point A to point C.

5.2 Volatility of returns

Note that volatility of returns can be written as

$$\mathbb{V}[v-p] = \mathbb{V}[\mathbb{E}(v-p|p)] + \mathbb{E}[\mathbb{V}[v-p|p]] \tag{11}$$

$$= \mathbb{V}[rp(p)] + \mathbb{E}[\mathbb{V}[v|p]] \tag{12}$$

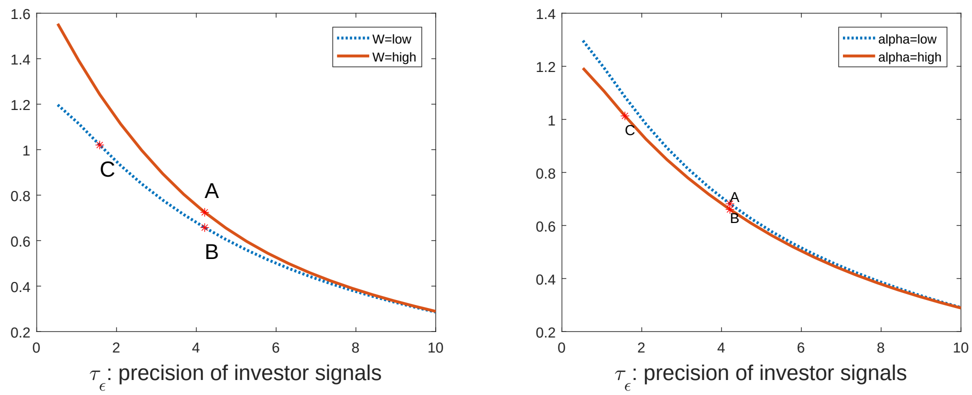
$$= \kappa_m^2 \mathbb{V}[x_m] + (\tau_v + \beta^2\tau_z)^{-1} \tag{13}$$

Because of the non-linearity of the conditional risk premium ($rp(p)$), the above expression cannot be simplified further. Hence, we proceed numerically.

Panel (a) of Figure 6 plots volatility of returns against τ_ϵ, precision of investors' signals for two different levels of wealth. Suppose point A is an equilibrium with high wealth level. Now imagine lowering wealth. We see that, holding τ_ϵ fixed, as wealth drops, volatility decreases (corresponding to the move from point A to point B). The intuition can be seen from equation 13. As wealth decreases, the second term does not change with fixed τ_ϵ. The first term decreases because of lower volatility of investors' (and hence market maker's) position. This is the direct effect. However, since investors acquire less information when they are constrained, we have

Figure 6: Price volatility

The figure plots price volatility as a function of precision of investors signal for different levels of wealth, $w = 0.2$ and $w = 2$ (panel (a)) and different levels of parameter α, $\alpha = 0.95$ and $\alpha = 0.99$ (panel (b)). Other parameter values are set to: $\tau_u = \tau_Z = \tau_v = \tau_e = 1$, $\bar{v} = 4, \gamma = 3, \kappa_m = 1.5$.



(a) (b)

an increase in volatility corresponding to the move from point B to point C. Therefore, the indirect direct operating through information spiral may dominate so that volatility increases as wealth drops.

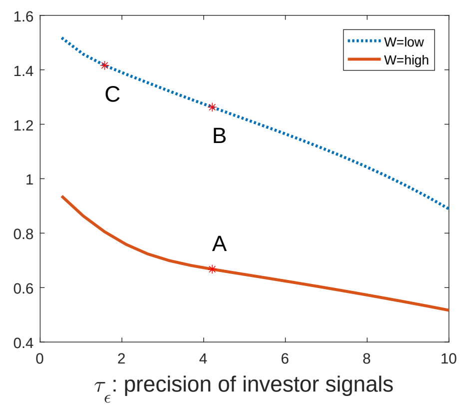
We proceed by examining the effects of margin requirements (as measured by parameter α) on price volatility. It has long been argued that tighter margin requirements should stabilize prices. The argument is that tighter margin requirements should curb the investors' positions therefore limiting the price impact of their information and liquidity shocks. Consider panel (b) of Figure 6. We see that as margin constraints tighten, volatility indeed drops as we move from point A to point B, confirming the above wisdom. This result holds with τ_ϵ fixed (direct effect). However, since investors acquire less information with tighter constraints, the volatility may increase with tighter funding requirements (we move from point B to point C). Thus, our model provides an alternative explanation for why tightening of margin requirements can increase volatility, complementing the results of Wang (2015).

5.3 Sharpe ratio

Finally, we examine the Sharpe ratio of the risky asset. The Sharpe ratio is defined as $SR = \frac{\mathbb{E}[v-p]}{\mathbb{V}[v-p]}$. With τ_ϵ fixed, we argued in the previous subsections that risk premium rises and volatility drops as wealth of investors decrease. This implies that Sharpe ratio rises as wealth drops, holding τ_ϵ fixed. Now, with endogenous learning, we argued in previous section that risk premium and volatility both rise. This implies that the indirect effect cannot be signed. In figure 7, we see that both direct effect(A to B) and indirect effect (B to C) are in the same direction, amplifying the effect of wealth shock.

Figure 7: Sharpe ratio

The figure plots Sharpe ratio as a function of precision of investors signal for different levels of wealth, $w = 0.2$ and $w = 2$. Other parameter values are set to: $\tau_u = \tau_Z = \tau_v = \tau_e = 1$, $\bar{v} = 4$,$\gamma = 3$,$\kappa_m = 1.5$.



6 Conclusion

In this paper we developed a tractable REE model with general portfolio constraints. We applied our methodology to study a model with endogenous margins constraints. We uncovered a novel amplification mechanism, which we call information spiral. A drop in investors' wealth tightens constraints and reduces their incentive to acquire information, which lowers price informativeness. Moreover, financiers who use information in prices to assess the risk of financing a trade face more uncertainty and set tighter margins, which further tightens constraints. This implies that risk premium, conditional volatility and Sharpe ratios rise disproportionately as investors' wealth drops. These results imply a new, information-based rationale why the wealth of investors is important.

Our information spiral can also potentially generate complementaries in information acquisition: as other investors acquire more information, margins become less tight, giving incentives to a particular investor to acquire more information. This provides a mechanism for complementaries in information acquisition alternative to that in Goldstein and Yang (2017). We explore implications of the above complementaries in the ongoing work.

While we have assumed normal distribution for the random variables in the model, our results can be generalized to distributions within the exponential family, as in Breon-Drish (2015). We leave it for the future work.

7 Appendix

Proof. (Proposition 1) We conjecture linear equilibrium. In such an equilibrium the price is informationally equivalent to a linear combination of v and z. In particular, we conjecture, and later verify that there exists a linear function $f(p) = f_0 + f_1 \cdot p$ and a constant β such that

$$f(p) = \phi, \text{ where } \phi = v - \beta^{-1} z.$$

We call β the *informational efficiency.*

In the conjectured equilibrium demand of the market maker and the aggregate demand of investors can be written as linear functions of p and ϕ. That is, there should exist constants c_0, c_p and c_ϕ and c_0^m, c_p^m and c_ϕ^m such that

$$X = c_0 + c_\phi \phi - c_p p, \;\; x_m = c_0^m + c_\phi^m \phi - c_p^m p, \text{ where } X = \int x_i di.$$

From the above and the market clearing condition it is easy to find

$$f_1 = \frac{c_p + c_p^m}{c_\phi + c_\phi^m}, \;\; f_0 = \frac{1 - c_0 - c_0^m}{c_\phi + c_\phi^m}.$$

It remains to find the coefficients c_p, c_ϕ, etc.. The first order conditions of an investor i and a market maker imply

$$x_i = \frac{\tau}{\gamma}(E[v|\mathcal{F}_i] - p) - e_i, \;\; x_m = \frac{1}{\kappa_m}(E[v|\mathcal{F}_m] - p), \tag{14}$$

where we introduced notation τ and τ_m for total precision of information available to an investor and market maker, respectively:

$$\tau = Var[v|\mathcal{F}_i]^{-1}, \;\; \tau_m = Var[v|\mathcal{F}_m]^{-1}.$$

Given our conjecture, the information set of market maker is $\mathcal{F}_m = \{\phi\}$, whereas the information set of an investor i is $\mathcal{F}_i = \{s_i, e_i, \phi\}$ and can be further simplified to $\mathcal{F}_i = \{s_i, \phi_i\}$, where

$$\phi_i = v - \beta^{-1}(z - E[z|e_i]) = v - \beta^{-1}\left(\frac{\tau_z z - \tau_u u_i}{\tau_u + \tau_z}\right).$$

Precisions of the signals ϕ and ϕ_i are given by

$$Var(\phi|v)^{-1} = \beta^2\tau_z, \;\; Var(\phi_i|v)^{-1} = \beta^2(\tau_z + \tau_u).$$

Projection Theorem implies that

$$\tau = \tau_v + \tau_\epsilon + \beta^2(\tau_z + \tau_u), \;\; \tau_m = \tau_v + \beta^2\tau_z.$$

$$E[v|\mathcal{F}_i] = \frac{\tau_v}{\tau}\bar{v} + \frac{\tau_\epsilon}{\tau}s_i + \frac{\beta^2(\tau_z + \tau_u)}{\tau}\phi_i, \;\; E[v|\mathcal{F}_m] = \frac{\tau_v}{\tau_m}\bar{v} + \frac{\beta^2\tau_z}{\tau_m}\phi.$$

Since the aggregate demand of investors and market makers can depend on v only through ϕ we find

$$c_\phi = \frac{\tau}{\gamma}\frac{\partial E[v|\mathcal{F}_i]}{\partial v} = \frac{\tau_\epsilon}{\gamma} + \frac{\beta^2(\tau_z + \tau_u)}{\gamma}, \;\; c_\phi^m = \frac{1}{\kappa_m}\frac{\partial E[v|\mathcal{F}_m]}{\partial v} = \frac{\beta^2\tau_z}{\kappa_m}.$$

Similarly,

$$c_0 = \frac{\tau_v}{\gamma}\bar{v}, \;\; c_0^m = \frac{\tau_v}{\kappa_m}\bar{v},$$

$$c_p = \frac{\tau}{\gamma}, \;\; c_p^m = \frac{1}{\kappa_m}.$$

I remains to pin down β. It is clear that in our equilibrium the ratio of sensitivities of $\frac{\partial x_i}{\partial v}$ to $\frac{\partial x_i}{\partial z}$ should be equal to β. Thus,

$$\beta = \frac{\tau_\epsilon + \beta^2(\tau_z + \tau_u)}{\beta\tau_z + \gamma}. \tag{15}$$

The above equation defines a fixed point problem to pin down equilibrium β. Provided that $\gamma^2 > 4\tau_\epsilon\tau_u$ this fixed-point problem admits two solutions (or no solutions otherwise), however only the smaller solution corresponds to a stable fixed point. The latter is given by:

$$\beta = \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{\tau_\epsilon}{\tau_u}}.$$

■

Proof. (Proposition 2) We prove that for every p there exists unique $\phi = f(p)$ such that market clears. Indeed, the market clearing can be written as

$$g(\phi, p) + c_0^m - c_p^m p + c_\phi^m \phi = 1.$$

For a given p aggregate investors' demand $g(\phi, p)$ is monotone in ϕ. Thus, there is at most one solution. At least one solution exists by the Intermediate Value Theorem. The aggregate demand at $+\infty(-\infty)$ is equal to $+\infty(-\infty)$, thus at some intermediate point aggregate demand has to be equal to 1.

We compute a closed-form expression for the aggregate demand of investors $g(\phi, p)$. It can be split into three parts. For a fraction π_1 of investors the lower constraint $a(p)$ will bind. The latter fraction can be calculated as follows

$$\pi_1(\phi, p) = Pr\left(X^u(p, \phi) + \xi_i < a(p)\right) = \Phi\left(\frac{a(p) - X^u(p, \phi)}{\sigma_\xi}\right),$$

where $\Phi(\cdot)$ denotes the CDF of a standard normal random variable. They will contribute $\pi_1(\phi, p)a(p)$ to the aggregate demand. Similarly, a fraction π_3 of investors for whom the upper constraint $b(p)$ binds will contribute $\pi_3(\phi, p)b(p)$, where

$$\pi_3(\phi, p) = 1 - \Phi\left(\frac{b(p) - X^u(p, \phi)}{\sigma_\xi}\right).$$

Finally a fraction $\pi_2(\phi, p) = 1 - \pi_1 - \pi_3$ will be unconstrained. They will contribute $\pi_2 \cdot (X^u +$

$E[\xi_i|(\xi_i + X^u) \in [a(p), b(p)]])$. The last term can be further simplified to

$$\pi_2 E[\xi_i|(\xi_i + X^u) \in [a(p), b(p)]] = \sigma_\xi \left(\Phi' \left(\frac{a(p) - X^u}{\sigma_\xi} \right) - \Phi' \left(\frac{b(p) - X^u}{\sigma_\xi} \right) \right).$$

Combining all of the terms we get

$$g(\phi, p) = \pi_1 a(p) + \pi_3 b(p) + \pi_2 X^u + \sigma_\xi \left(\Phi' \left(\frac{a(p) - X^u}{\sigma_\xi} \right) - \Phi' \left(\frac{b(p) - X^u}{\sigma_\xi} \right) \right).$$

■

Proof. (Proposition 3) The certainty equivalent at time 1 is given by

$$CE_1 = \begin{cases} e_i p + \frac{\gamma}{2\tau_i} \left((x_i^u + e_i)^2 - (x_i^u - a)^2 \right) & \text{if} \quad x_i^u < a \\ e_i p + \frac{\gamma}{2\tau_i} \left((x_i^u + e_i)^2 \right) & \text{if} \quad b > x_i^u > a \\ e_i p + \frac{\gamma}{2\tau_i} \left((x_i^u + e_i)^2 - (x_i^u - b)^2 \right) & \text{if} \quad x_i^u > b \end{cases}$$

Define $c_i = \sqrt{\tau_i} \mathbb{E}_i(v - p)$. Then the certainty equivalent at time 1 can be rewritten by

$$CE_1 = \begin{cases} e_i p + \frac{c_i(e_i + a)}{\sqrt{\tau_i}} - \gamma \frac{(e_i + a)^2}{2\tau_i} & \text{if} \quad c_i < \frac{\gamma}{\sqrt{\tau_i}} (a + e_i) \\ e_i p + \frac{c_i^2}{2\gamma} & \text{if} \quad \frac{\gamma}{\sqrt{\tau_i}} (b + e_i) > c_i > \frac{\gamma}{\sqrt{\tau_i}} (a + e_i) \\ e_i p + \frac{c_i(e_i + b)}{\sqrt{\tau_i}} - \gamma \frac{(e_i + b)^2}{2\tau_i} & \text{if} \quad c_i > \frac{\gamma}{\sqrt{\tau_i}} (b + e_i). \end{cases}$$

Therefore, the optimal ex-ante utility at time 0 for investor i can be written as

$$U_0(\tau_{ei}, \tau_e) = \mathbb{E}[h(e_i, p) U_1(\tau_{ei}, \tau_e, e_i, p)]$$

where we define

$$U_1(\tau_{ei}, \tau_e, e_i, p) = -\mathbb{E}(\exp(-\gamma CE_1 + \gamma e_i p)|e_i, p)$$

and $h(e_i, p) = \exp(-\gamma e_i p)$. Decompose $U_1(\tau_{ei}, \tau_e, e_i, p)$ into regions where the lower constraint binds, unconstrained and where the upper constraint binds,

$$= \begin{cases} -\exp\left(\gamma^2 \frac{(e_i+a)^2}{2\tau_i}\right) \int\limits_{-\infty}^{\frac{\gamma}{\sqrt{\tau_i}}(a+e_i)} \exp\left(-\frac{c_i(e_i+a)}{\sqrt{\tau_i}}\right) dF(c_i) & \equiv U_1^a(\tau_{ei},\tau_e,e_i,p) \\ -\int\limits_{\frac{\gamma}{\sqrt{\tau_i}}(a+e_i)}^{\frac{\gamma}{\sqrt{\tau_i}}(b+e_i)} \exp\left(-\frac{c_i^2}{2}\right) dF(c_i) & \equiv U_1^u(\tau_{ei},\tau_e,e_i,p) \\ -\exp\left(\gamma^2 \frac{(e_i+b)^2}{2\tau_i}\right) \int\limits_{\frac{\gamma}{\sqrt{\tau_i}}(a+e_i)}^{\infty} \exp\left(-\frac{c_i(e_i+b)}{\sqrt{\tau_i}}\right) dF(c_i) & \equiv U_1^b(\tau_{ei},\tau_e,e_i,p) \end{cases}$$

We can write

$$U_1(\tau_{ei},\tau_e,e_i,p) = U_1^a(\tau_{ei},\tau_e,e_i,p) + U_1^u(\tau_{ei},\tau_e,e_i,p) + U_1^b(\tau_{ei},\tau_e,e_i,p)$$

Define $\sigma_i^2 = Var_i\,(c_i|e_i,p)$ and $\mu_i = E_i\,(c_i|e_i,p)$. In lemma 3, we prove that $\frac{\mu_i^2}{1+\sigma_i^2}$ is independent of investor i's information acquisition. Lets denote it by $\psi(e_i,p)$. Applying lemma 4, we can simplify $U_1^a(\tau_{ei},\tau_e,e_i,p)$ and $U_1^b(\tau_{ei},\tau_e,e_i,p)$ as

$$\begin{aligned} U_1^a(\tau_{ei},\tau_e,e_i,p) = -\exp&\left(\frac{\gamma^2\,(e_i+a)^2}{2\,(\tau_v+\beta^2\,(\tau_u+\tau_z))} - E\,(v-p|p,e_i)\,(e_i+a)\,\gamma\right) \\ &\Phi\left(\sqrt{\frac{\tau_i}{\tau_{ei}}}\left(\frac{\gamma\,(a+e_i)-(\tau_v+\beta^2\,(\tau_u+\tau_z))\,E\,(v-p|p,e_i)}{\sqrt{(\tau_v+\beta^2\,(\tau_u+\tau_z))}}\right)\right) \end{aligned} \tag{16}$$

$$\begin{aligned} U_1^b(\tau_{ei},\tau_e,e_i,p) = -\exp&\left(\frac{\gamma^2\,(e_i+b)^2}{2\,(\tau_v+\beta^2\,(\tau_u+\tau_z))} - E\,(v-p|p,e_i)\,(b+e_i)\,\gamma\right) \\ &\Phi\left(\sqrt{\frac{\tau_i}{\tau_{ei}}}\frac{E\,(v-p|p,e_i)\,(\tau_v+\beta^2\,(\tau_u+\tau_z))-\gamma\,(b+e_i)}{\sqrt{(\tau_v+\beta^2\,(\tau_u+\tau_z))}}\right) \end{aligned} \tag{17}$$

Note that the first exponent is independent of investor i's information choice. Applying lemma 4, we can simplify $U_1^u(\tau_{ei}, \tau_e, e_i, p)$ as

$$
\begin{aligned}
&U_1^u(\tau_{ei}, \tau_e, e_i, p) \\
&\quad = -\sqrt{\frac{\tau_v + \beta^2(\tau_u + \tau_z)}{\tau_i}} \exp\left(-\frac{\psi(e_i, p)}{2}\right) \left(\Phi\left(\frac{\gamma(a + e_i) - (\tau_v + \beta^2(\tau_u + \tau_z)) E(v - p|p, e_i)}{\sqrt{\tau_{ei}}}\right)\right. \\
&\qquad \left. - \Phi\left(\frac{\gamma(a + e_i) - (\tau_v + \beta^2(\tau_u + \tau_z)) E(v - p|p, e_i)}{\sqrt{\tau_{ei}}}\right)\right)
\end{aligned} \tag{18}
$$

Using the above definitions, we can split investors time 0 utility as $U_0(\tau_{ei}, \tau_e) = U_0^a(\tau_{ei}, \tau_e) + U_0^u(\tau_{ei}, \tau_e) + U_0^b(\tau_{ei}, \tau_e)$ where we define

$$U_0^a(\tau_{ei}, \tau_e) = \mathbb{E}[h(e_i, p)U_1^a(\tau_{ei}, \tau_e, e_i, p)]$$

$$U_0^u(\tau_{ei}, \tau_e) = \mathbb{E}[h(e_i, p)U_1^u(\tau_{ei}, \tau_e, e_i, p)]$$

$$U_0^b(\tau_{ei}, \tau_e) = \mathbb{E}[h(e_i, p)U_1^b(\tau_{ei}, \tau_e, e_i, p)]$$

Next, we compute the marginal value of information:

$$\frac{\partial U_0(\tau_{ei}, \tau_e)}{\partial \tau_{ei}} = \mathbb{E}\left[h(e_i, p)\left(\frac{\partial U_1^a(\tau_{ei}, \tau_e, e_i, p)}{\partial \tau_{ei}} + \frac{\partial U_1^u(\tau_{ei}, \tau_e, e_i, p)}{\partial \tau_{ei}} + \frac{\partial U_1^b(\tau_{ei}, \tau_e, e_i, p)}{\partial \tau_{ei}}\right)\right]$$

After a lot of tedious algebra, we can show that

$$
\begin{aligned}
\frac{\partial U_0(\tau_{ei}, \tau_e)}{\partial \tau_{ei}} &= -\frac{1}{2\tau_i}\mathbb{E}\left[h(e_i, p)U_1^u(\tau_{ei}, \tau_e, e_i, p)\right] \\
&= -\frac{1}{2\tau_i}U_0^u(\tau_{ei}, \tau_e)
\end{aligned}
$$

We can write the investor's problem as

$$\max_{\tau_e i > 0} U_0(\tau_{ei}, \tau_e)\exp(\gamma C(\tau_{ei})).$$

The first order condition for this problem is

$$
\begin{aligned}
C'(\tau_{ei}) &= -\frac{1}{U_0(\tau_{ei},\tau_e)}\frac{\partial U_0(\tau_{ei},\tau_e)}{\partial \tau_{ei}} \\
&= \frac{1}{2\tau_i}\frac{U_0^u(\tau_{ei},\tau_e)}{U_0(\tau_{ei},\tau_e)}
\end{aligned}
$$

■

Proof. (Proposition 4) We write the expression for the marginal value of information MVI $= \frac{1}{2\tau_i}\frac{U_0^u}{U_0}$. The only term affected by constraints is $\frac{U_0^u}{U_0}$. Consider first the nominator: $U_0^u = E[-e^{-\gamma CE_1}\mathbb{I}(x_i^u = x_i)]$. It increases (becomes less negative) as constraints become tighter: recall that investors get negative utility; as constraints become tighter, they get it in fewer states of the world. The denominator U_0 decreases (becomes more negative) as with constraints the certainty equivalent CE_1 in all states weakly decreases. Thus, the ratio decreases. In the case of risk-neutral market maker with no inventory costs, constraints do not alter prices, therefore the same argument applies when for all investors' constraints become tighter. ■

Lemma 3. *Define $\sigma_i^2 = Var_i\left(c_i|e_i,p\right)$ and $\mu_i = E_i\left(c_i|e_i,p\right)$. Then the following results hold:*

- $\frac{\mu_i^2}{1+\sigma_i^2}$ *is independent of investor i's information acquisition.*
- σ_i *satisfies*

$$\sigma_i^2 = \frac{\tau_{ei}}{\tau_v + \beta^2\left(\tau_u + \tau_z\right)}$$

Lemma 4. *If x is normally distributed with mean μ and variance σ, then*

- $$\int_l^m exp\left(-\frac{x^2}{2}\right)dF(x) = \frac{1}{\sqrt{1+\sigma^2}}exp\left(-\frac{1}{2}\frac{\mu^2}{1+\sigma^2}\right)\left(\Phi\left(\frac{m-\frac{\mu}{1+\sigma^2}}{\sqrt{\frac{\sigma^2}{1+\sigma^2}}}\right) - \Phi\left(\frac{l-\frac{\mu}{1+\sigma^2}}{\sqrt{\frac{\sigma^2}{1+\sigma^2}}}\right)\right)$$

- $$\int_l^\infty exp\left(-kx\right)dF(x) = exp\left(\frac{k^2\sigma^2}{2} - \mu k\right)\Phi\left(\frac{\mu - l - k\sigma^2}{\sigma}\right)$$

Proof. (Proposition 6) One can prove that for every p there exists unique $\phi = f(p)$ such that market clears similarly to Proposition 2.

We now prove that $f(p)$ is invertible. We plug expression for our endogenous margins into ODE (3) assuming that both m_n^+ and m_n^- are positive. We get

$$f'(p) = \frac{c_p^m + \pi_2 c_p - \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2}\right) rp(p)'}{\pi_2 c_\phi + c_\phi^m}.$$

from which, accounting for (10) we find

$$f'(p) = \frac{c_p^m + \pi_2 c_p + \kappa_m \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2}\right) c_p^m}{\pi_2 c_\phi + c_\phi^m + \kappa_m \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2}\right) c_\phi^m}.$$

Both m_n^+ and m_n^- are positive, when $f(p) \in \left[-\Phi^{-1}(\alpha)\frac{\sigma_{v|p}}{c_\phi^m}\frac{1}{\kappa_m} - \frac{c_0^m}{c_\phi^m} + \frac{c_p}{c_\phi^m}p; \Phi^{-1}(\alpha)\frac{\sigma_{v|p}}{c_\phi^m}\frac{1}{\kappa_m} - \frac{c_0^m}{c_\phi^m} + \frac{c_p}{c_\phi^m}p\right] = [f^-(p); f^+(p)]$ Proceeding similarly, one can get

$$f'(p) = \begin{cases} \frac{c_p^m + \pi_2 c_p + \frac{1}{\kappa_m}\frac{\pi_3 W_0}{m^+(p)^2} c_p^m}{\pi_2 c_\phi + c_\phi^m + \frac{1}{\kappa_m}\frac{\pi_1 W_0}{m^-(p)^2} c_\phi^m}, & \text{if } f < f^-(p), \\ \frac{c_p^m + \pi_2 c_p + \frac{1}{\kappa_m}\left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2}\right) c_p^m}{\pi_2 c_\phi + c_\phi^m + \frac{1}{\kappa_m}\left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2}\right) c_\phi^m}, & \text{if } f^-(p) < f < f^+(p), \\ \frac{c_p^m + \pi_2 c_p + \frac{1}{\kappa_m}\frac{\pi_1 W_0}{m^-(p)^2} c_p^m}{\pi_2 c_\phi + c_\phi^m + \frac{1}{\kappa_m}\frac{\pi_1 W_0}{m^-(p)^2} c_\phi^m}, & \text{if } f > f^+(p). \end{cases}.$$

Clearly, the derivative above is always positive, which means that the equilibrium function $f(p)$ is invertible. Thus, for each fundamental ϕ there exists a unique p clearing the market. The initial condition for the ODE above can be found by clearing the market for a particular price, e.g., price $p = 0$.

■

Proof. (Lemma 2) We show that given aggregate state (v, z) when W_0 increases, maximal short and long positions $-a(v, z)$ and $b(v, z)$ increase. We prove this by contradiction. For convenience, we restate $a(v,z)$ and $b(v,z)$ here

$$a(v,z) = -\frac{W_0}{\left[\Phi^{-1}(\alpha)/\sqrt{\tau_m} + rp(v,z)\right]^+}, \quad b(v,z) = \frac{W_0}{\left[\Phi^{-1}(\alpha))/\sqrt{\tau_m} - rp(v,z)\right]^+}$$

In what follows, we do not write the argument (v,z) explicitly. Recall that in the absence of information acquisition, a shock on W_0 affects margin only through risk premium rp. First, we show that a and b cannot both (weakly) increase or decrease. Suppose contrary that both a and b increases when W_0 increases. The aggregate demand from investors increases and the demand from market maker x_m decreases. An increased (less negative) a when also implies a strictly increased m^- and thus an increased rp. This contradicts with the fact that x_m decreases because $x_m = \kappa_m rp$. A symmetric argument can show that a and b cannot both (weakly) decrease.

Second, when W_0 increases, it cannot be the case that a increases and b decreases. This is because a decrease in b implies a strict decrease in rp, which in turn implies a strict decrease in a, i.e. a contradiction.

In sum, the only remaining possibility is that a decreases and b increases when W_0 increases. ■

Proof. (Proof of Proposition 7) Follows by plugging $rp = 0$ to equations (8) and(9). ■

Appendix-B: Application to Yuan (2005)

In this appendix, we apply our methodology to borrowing constraints studied in Yuan (2005). In this case, Borrowing-constrained informed investor demand is bounded above by $b(p) = \delta_0 + \delta_1 p$ where $\delta_1 > 0$ and there is no lower bound on investor demand.

The borrowing constraint is a function of the price. The lower the asset price, the harder it is for informed investors to raise outside financing to invest in the risky asset.

$$f'(p) = \frac{c_p^m + (1-\pi_3)c_p - \pi_3\delta_1}{(1-\pi_3)c_\phi + c_\phi^m}$$

where all the coefficients are positive and π_3 denotes the mass of investors for which the constraint binds. The following theorem gives conditions under which there will be multiple equilibrium.

Proposition 8. *When the constraint is of the form $b(p) = \delta_0 + \delta_1 p$ where $\delta_1 > 0$, equilibrium is unique when $\delta_1 < \frac{1}{\kappa_m}$ and there will be multiple equilibria otherwise.*

Proof. (Proposition 8) In this case,

$$f'(p) = \frac{c_p^m + (1-\pi_3)c_p - \pi_3\delta_1}{(1-\pi_3)c_\phi + c_\phi^m} \tag{19}$$

where π_3 denotes the mass of investors for which the constraint binds. As p decreases, π_3 increases and numerator of equation (19) increases. In the extreme case, as p tends to low number, most of informed investors are binding and numerator tends to $c_p^m - \delta_1$. If this term is positive, we will always have unique equilibrium because $f'(p) > 0 \forall p$. If this term becomes negative, there could be multiple equilibria. ■

What we mean by multiple equilibrium is that some realization of fundamentals can be supported by two prices. This results from the interaction of substitution and information effects in the model. In typical REE model (Hellwig (1980b)), the substitution effect always dominates the information effect leading to unique equilibrium. In those models, the information effect is fixed as prices reveal the same amount of information regardless of level. In our setting, due to the borrowing constraint imposed on informed investors, unit change in price does not reflect the same information. This implies that information effect can dominate substitution effect for some realization of prices and there will be multiple equilibrium.

Bibliography

Philip Bond, Alex Edmans, and Itay Goldstein. The real effects of financial markets. The Annual Review of Financial Economics is, 4:339–60, 2012.

B. Breon-Drish. On existence and uniqueness of equilibrium in noisy rational expectations economies. The Review of Economic Studies, 82(3):868–921, 2015.

Markus K Brunnermeier and Lasse Heje Pedersen. Market liquidity and funding liquidity. Review of Financial Studies, 22(6):2201–2238, 2009.

Markus K Brunnermeier and Yuliy Sannikov. A macroeconomic model with a financial sector. The American Economic Review, 104(2):379–421, 2014.

Douglas W Diamond and Robert E Verrecchia. Information aggregation in a noisy rational expectations economy. Journal of Financial Economics, 9(3):221–235, 1981.

James Dow, Itay Goldstein, and Alexander Guembel. Incentives for information production in markets where prices affect real investment. Journal of the European Economic Association, page jvw023, 2017.

Ana Fostel and John Geanakoplos. Endogenous collateral constraints and the leverage cycle. Annu. Rev. Econ., 6(1):771–799, 2014.

Jayant Vivek Ganguli and Liyan Yang. Complementarities, multiplicity, and supply information. Journal of the European Economic Association, 7(1):90–115, 2009.

Itay Goldstein and Liyan Yang. Information disclosure in financial markets. Annual Review of Financial Economics, 2017.

Itay Goldstein, Emre Ozdenoren, and Kathy Yuan. Trading frenzies and their impact on real investment. Journal of Financial Economics, 109(2):566–582, 2013.

Denis Gromb and Dimitri Vayanos. Equilibrium and welfare in markets with financially constrained arbitrageurs. Journal of financial Economics, 66(2):361–407, 2002.

Denis Gromb and Dimitri Vayanos. The dynamics of financially constrained arbitrage. Technical report, National Bureau of Economic Research, 2017.

Sanford Grossman. On the efficiency of competitive stock markets where trades have diverse information. The Journal of finance, 31(2):573–585, 1976.

S.J. Grossman and J.E. Stiglitz. On the impossibility of informationally efficient markets. The American Economic Review, 70(3):393–408, 1980.

Zhigu He and Arvind Krishnamurthy. A model of capital and crises. The Review of Economic Studies, 79(2):735–777, 2011.

Zhiguo He and Arvind Krishnamurthy. Intermediary asset pricing. The American Economic Review, 103(2):732–770, 2013.

Martin Hellwig. On the aggregation of information in competetive markets. Journal of Economic Theory, 3:477–498, 1980a.

Martin F Hellwig. On the aggregation of information in competitive markets. Journal of economic theory, 22(3):477–498, 1980b.

Albert S Kyle and Wei Xiong. Contagion as a wealth effect. The Journal of Finance, 56(4): 1401–1440, 2001.

Carolina Manzano and Xavier Vives. Public and private learning from prices, strategic substitutability and complementarity, and equilibrium multiplicity. Journal of Mathematical Economics, 47(3):346–369, 2011.

Mahdi Nezafat, Mark Schroder, and Qinghai Wang. Short-sale constraints, information acquisition, and asset prices. Journal of Economic Theory, 172:273–312, 2017.

Lin Peng and Wei Xiong. Investor attention, overconfidence and category learning. Journal of Financial Economics, 80(3):563–602, 2006.

Stijn Van Nieuwerburgh and Laura Veldkamp. Information immobility and the home bias puzzle. The Journal of Finance, 64(3):1187–1215, 2009.

Stijn Van Nieuwerburgh and Laura Veldkamp. Information acquisition and under-diversification. Review of Economic Studies, 77(2):779–805, 2010.

Yajun Wang. Why can margin requirements increase volatility and benefit margin constrained investors? Review of Finance, 20(4):1449–1485, 2015.

Wei Xiong. Convergence trading with wealth effects: an amplification mechanism in financial markets. Journal of Financial Economics, 62(2):247–292, 2001.

Kathy Yuan. Asymmetric price movements and borrowing constraints: A rational expectations equilibrium model of crises, contagion, and confusion. The Journal of Finance, 60(1):379–411, 2005.