Strategic Trading without Normality

Sergei Glebkin*

Abstract

I present a model of strategic trading a la Kyle (1989) that does not require the assumption of normally distributed asset payoffs. I propose a constructive solution method: finding the equilibrium reduces to solving a linear ordinary differential equation. With non-normal payoffs, the price response becomes an asymmetric, non-linear function of order size: greater for buys than sells and concave (convex) for small sell (buy) orders when asset payoffs are positively skewed; concave for large sell (buy) orders when payoffs are bounded below (above). The model can speak to key empirical findings and provides new predictions concerning the shape of price impact.

1 Introduction

In many markets trade is dominated by large institutional investors (such as mutual and pension funds) whose trades can affect prices. These investors often trade strategically, taking their price impact into account.[1] Empirical evidence documents that prices react to orders of such investors in an asymmetric and non-linear way: purchases typically have greater price impact compared to sells and price response is a concave function of order size.[2]

Previous papers on strategic trading have often adopted a CARA-normal framework for tractability: traders have negative exponential (CARA) utility functions and asset payoffs are normally distributed. The CARA-normal models feature linear equilibria (in which the price is a linear function of order size and purchases and sells have the same price impact) which are hard to align with empirical evidence. Normality also implies that higher moments play no role which may not be true in practice, and that asset payoffs are not bounded which is unrealistic, e.g., due to limited liability. In this paper I present a tractable model of strategic trading that allows for general distribution of asset payoffs. I show that if asset payoffs are positively skewed and bounded the model can speak to key empirical findings

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[1] Some investors, such as J.P. Morgan or Citigroup, have in-house optimal execution desks which devise trading strategies to minimize price-impact costs. Other investors use the software and services provided by more specialized trading firms.

[2] Hausman et al. (1992), Almgren et al. (2005), Frazzini et al. (2014) find concave price reaction functions (absolute value of price change as a function of order size) for equities. Muraviev (2015) presents the evidence for options. He decomposes the price reaction function into inventory and information components and finds that both are concave. Regarding asymmetry, Saar (2001) summarizes the evidence that shows bigger price impact of buy orders compared to sell orders. However, Chiyachantana et al. (2004) link the asymmetry to the underlying market condition and find that in bullish markets buy orders have a bigger price impact than sells, while in the bearish markets sells have a higher price impact.

concerning nonlinearity and asymmetry of price impact. The main technical challenge is that with non-normal distribution the traditional guess-and-verify approach is no longer applicable as it is not clear what should be the guess. I propose a constructive solution method that allows to overcome this difficulty and to solve the model in closed form for any distribution.

I assume that CARA traders exchange a risky for a riskless asset over one period. Traders have the same risk aversion coefficient and are symmetrically informed. Trading is structured as a uniform-price double auction: traders submit simultaneously demand functions, and all trades are executed at the price that clears the market. My main innovation relative to previous literature is to assume that the distribution of the risky asset payoff is completely general save for the technical restriction. The restriction requires that a *risk function*, a transformation of the cumulant generating function (CGF) which I introduce in this paper, exists.[3] This restriction holds for any distribution with finite support as well as for many infinite support distributions, including normal and mixture of normals. In addition to the symmetric CARA traders, there is a "block trader" who submits an exogenous market order of random size. Trade occurs because the CARA traders compete to absorb part of the block trader's order, hence providing liquidity to that trader. In most of the paper I assume that the block trader's order is independent of the asset payoff, and so the block trader is uninformed. In that setting the unique source of price impact is inventory risk. I also extend my model to allow the block trader's order to be correlated with the asset payoff. In that extension, price impact is driven by both inventory risk and asymmetric information. The model is similar to Kyle (1989), with the main simplification of absence of heterogeneity among strategic traders (both in terms of information and risk aversion) and the main generalization of allowing for non-normal payoffs.

In equilibrium, traders determine their optimal demand function knowing the demand functions of all other traders. I show that the optimization problem is equivalent to traders not knowing others' demand functions but knowing their own price impact (i.e., how their trade moves the price at the margin) for each order size. This is an intuitive representation of the problem: real-world traders typically have a market impact model that is an input in their optimal execution algorithm. The equilibrium price impact function is pinned down by the requirement that it is consistent with the demand functions of the other traders. The consistency requirement yields a linear ordinary differential equation (ODE) that I use to compute the equilibrium price impact function in closed form for any probability distribution. I show that the properties of the price impact function can be derived by those of a risk function, which summarizes the probability distribution of the asset payoffs. I derive main properties of a risk function, in particular the ones related to comparative statics, and believe that these can be useful for future research.

I show that the ODE for the price impact function can have a continuum of solutions, even with a normal distribution. Thus, there is a continuum of equilibria. Equilibrium non-uniqueness can be attributed to the following complementarity: if strategic traders believe that the price impact is high, they provide less liquidity, which confirms higher equilibrium price impact. In Glebkin, Rostek and Yoon (2015) we study equilibrium uniqueness in demand functions. Applying the results from the latter paper, a unique equilibrium with bounded payoff can be pinned down by requiring prices to lie within asset payoff bounds. This requirement is intuitive: if the price is outside payoff bounds the

[3]By definition, the cumulant generating function (CGF) is a log of a moment generating function of the distribution. Given the CGF $g(x)$ the risk function with a parameter a is defined as $\rho_a(x) = \int_0^1 g''(-t^{1-a}\gamma x)dt$.

block trader gets negative profit with certainty and hence should not trade. The requirement rules out equilibria in which CARA traders' price impact is too high, selecting the unique equilibrium. With unbounded payoff the equilibrium can be selected by requiring the prices to be close to that of an asset with an arbitrary close, but bounded payoff. This equilibrium corresponds to the linear one under normality.

Using the characterization of the price impact function, I examine the relationship between price and order size. When asset payoffs are positively skewed, small purchases have greater price impact compared to small sells. The intuition for the result can be seen by contrasting with the benchmark case where the asset payoff is normally distributed and hence the skewness is zero. Consider first sells by the block trader. The trader's counterparties, who buy from him, receive a positively skewed profit, which they like. Intuitively, positive skewness implies that positive surprises to profits are more likely than negative ones. As a result, traders require a lower premium for providing liquidity and the price reaction to the order is smaller than in the benchmark case. For purchases, the trader's counterparties, who sell to him, receive a negatively skewed profit and require a greater premium. The price reaction is greater than in the benchmark case. Consequently, with positive skewness small purchases have greater price impact compared to small sells. Similarly, when asset payoffs are negatively skewed sells have greater price impact compared to purchases.

The concavity of the price reaction function (absolute value of price change as a function of order size) for large orders arises when asset payoffs are bounded. If asset payoff is bounded below, e.g., by zero, the sell order cannot push the price below zero. Consequently, the price reaction function for sells is bounded above, which rules out convex shapes. Assuming a further mild restriction on payoff distribution I show that the price reaction function is concave for large sell orders.[4] Similarly, if payoff distribution is bounded above and satisfies the same restriction, the price reaction function is concave for large buy orders.

With positive skewness the price reaction function is concave for small sells and convex for small purchases. As noted above, with positive skewness the price reaction for purchases is smaller than that in a benchmark CARA-normal case, in which the price reaction linear. This "smaller than linear" price reaction generates a concave shape. For purchases, the price response is greater than that in a benchmark case, which generates convexity. Similarly, with negative skewness, the price reaction function is concave for small purchases and convex for small sells.

Summarizing, the baseline model can speak to two key empirical findings concerning the shape of the price impact: the asymmetry and the concavity of the price reaction function. With positively skewed asset payoffs the model predicts that price impact of (small) purchases is greater than that of (small) sells, which is in line with the evidence summarized by Saar (2001). With bounded asset payoffs the model predicts concave price reaction function for large orders, consistent with the evidence in Hausman et al. (1992), Almgren et al. (2005) and Frazzini et al. (2015). I also derive new predictions. The model implies that difference in curvatures of price reaction function for small purchases and sells is positive (negative) with positive (negative) skewness. This prediction can be tested in equities market: individual stocks have positively skewed returns, while returns on stock indices are negatively skewed

[4] The restriction is that the third derivative of the CGF, $g'''(x)$, does not change sign for large enough x. In the numerical simulations (as well as analytic calculations, when direct calculation of CGF is possible) I was unable to find an example of a finite-support distribution for which this condition does not hold. Consequently, I believe that this restriction is mild.

(e.g. Chen et al. (2001)). The effects of payoff bounds can be examined in options market. Payoffs of puts, unlike that of calls are bounded above. The model suggests that the price reaction to purchases should be more concave for puts rather than calls.

In the extension of the model I show that when the block trader possesses private information regarding the asset payoff, the price impact can be separated into an inventory risk part and asymmetric information one. The bounds of the asset payoff play the same role as in the case of no asymmetric information. Unlike that case, however, there are additional determinants of the curvature of price reaction function for small orders. In particular, the shape of the conditional moments (mean and variance of asset payoff conditional on order size of the block trader) as functions of order size also plays a role. For example, if (1) skewness is positive, (2) the conditional mean function is convex and (3) the conditional variance is decreasing, both inventory and asymmetric information components of the price reaction function are convex (concave) functions of order size for small buy (sell) orders. One would not capture these effects in the jointly normal setting because, in that case, the conditional mean is linear and the conditional variance is a constant.

In the extended setting, I also show that the slope of the information component of the price reaction function is diagnostic of the degree of informed trading. This slope is proportional to the slope of the conditional mean function. The latter shows how a marginal unit liquidated or purchased by a block trader affects expectation of liquidity providers regarding asset payoffs. Therefore, a higher slope of the price reaction function indicates a more informed block trader.

This paper is related to three broad strands of the literature: strategic trading, divisible good auctions, and models of asset trading without normality.

The literature on strategic trading dates back to Kyle (1985). The models typically feature asymmetric information and rely on CARA-normal assumptions for tractability.[5] The equilibria in those models are linear, which results in liquidity measures being linear in the trade size.

The representation of equilibrium in this paper and the corresponding intuition builds on the result of Rostek and Weretka (2015), who are the first to show that the Nash equilibrium in demand submission games can be represented through two conditions: the optimality of the bid given a price impact model and the consistency of the model. A related paper by Weretka (2011) defines a new equilibrium concept in which traders are not price takers but slope takers. The consistency in that paper is a part of that equilibrium concept. In both Rostek and Weretka (2015) and Weretka (2011), however, the price impact is a constant, which is due to normality in the first paper and is a part of the equilibrium concept in the second paper.

A number of papers seek to explain the shape of the price impact. Rosu (2009) provides a model of the limit order book in which the key friction is costs associated with waiting for the execution of the limit orders. Keim and Madhavan (1996) explain concave price impact through a search friction in the upstairs market for block transactions. Saar (2001) provides an institutional explanation for the price impact asymmetry across buys and sells. My paper adds to this literature by providing a unified treatment of the properties of the price reaction function and linking them to the shape of the probability distribution that describes asset payoffs.

[5]An incomplete list includes: Kyle (1985, 1989), Subrahmanyam (1991), Douglas and Viswanathan (1996), Vayanos (1999, 2001), DeMarzo and Urosevic (2006), Rostek and Weretka (2015). See also Brunnermeier (2001) or Biais et al. (2005) for a review.

The divisible good auctions literature commonly examines the two most popular auction formats, namely the discriminatory-price auction (DPA) and the uniform-price auction (UPA), that are used to distribute divisible goods such as government debt, electricity, spectrum and emission permits.[6] The key contribution of my paper relative to this literature is to examine nonlinear equilibrium in UPA, solve for equilibrium in closed form and to link the nonlinearities to the properties of asset payoff distribution.

Wang and Zender (2002) consider both UPA and DPA formats in a CARA-normal setting with random supply of the asset. They show that there is a continuum of equilibria when uniform-price auction format is adopted. My non-uniqueness result is thus related to the one in that paper. My main contribution relative to Wang and Zender (2002) is to consider a more general setting allowing, in particular, the asset payoff to be bounded. The latter enables me to use the selection argument resulting in a unique equilibrium.

DeMarzo and Skiadas (1998, 1999), Breon-Drish (2015), Chabakauri et al. (2015) and Albagli, Hellwig, and Tsyvinski (2011) study REE models without assuming normality. Palvolgyi and Venter (2015) demonstrate the existence of non-linear equilibria with discontinuous price function in a standard CARA-normal REE model a la Grossman and Stiglitz (1980) and Hellwig (1980). The traders in those papers are competitive. Consequently, these papers abstract from price impact, which is the focus of my paper. Rochet and Vila (1994) analyze a model a la Kyle (1985) without normality and prove uniqueness of equilibrium.[7] Their uniqueness result is in contrast to the multiplicity result in my paper, which extends the Kyle (1989) model. Moreover, Rochet and Vila (1994) do not study how departures from normality affect the shapes of the price impact, which I do. Biais et al. (2000), Baruch (2005) and Back and Baruch (2013) study strategic liquidity provision without assuming normality. However, the liquidity providers in their models are risk-neutral. Consequently, there is no inventory risk, which is the main focus of my paper. Martin (2013) introduces the language of cumulant generating functions (CGF) in a Lucas tree asset pricing model with general (not log-normal) i.i.d. consumption growth. This paper applies the language of CGFs and demonstrates that it is useful in a strategic setting.

Finally, my paper speaks to a literature on optimal optimal dynamic execution algorithms under exogenous and non-constant price impact (see, e.g., Almgren et al. (2005)).[8] My paper complements this literature by providing equilibrium foundations for non-linear price functions.

The remainder of the paper is organized as follows. Section 2 presents the model, and Section 3 solves for equilibrium. Section 4 derives the model's predictions about the price impact measures and solves the model for specific probability distributions. Section 5 extends the model to the case where the block trader is informed. Section 6 links the theory to empirical evidence. Section 7 concludes. All proofs are in Appendix A. Appendix B collects the main properties of the risk functions.

[6]Such papers include Wilson (1979), Klemperer and Meyer (1989), Wang and Zender (2002), Kremer and Nyborg (2004), Rostek and Weretka (2012) and Ausubel et al (2014). The papers focusing specifically on finance applications, such as Kyle (1989), Vayanos (1999) and Rostek and Weretka (2015) (which I mentioned in the strategic trading literature) also belong to this literature.

[7]Boulatov and Bernhardt (2015) demonstrate the uniqueness of a *robust* equilibrium in a Kyle (1983) paper. They also derive, using a different technique, an ODE for price impact in their model. Their model features risk-neutral market makers and normally distributed asset payoff.

[8]Other papers on optimal dynamics execution algorithms with exogenous price impact, which can be linear or non-linear, include Bertsimas and Lo (1998), Almgren and Chriss (2001), Huberman and Stanzl (2005) and Obizhaeva and Wang (2012)

2 The model

There are two time periods $t \in \{0, 1\}$; two assets, a stock and a bond; and $L > 2$ large traders. The bond earns zero net interest, without loss of generality, as the bond can be chosen as a numeraire. The stock is a claim on a terminal dividend δ characterized by the *cumulant generating function (CGF)*

$$g(x) \equiv \log E[\exp(x\delta)].$$

I assume that the CGF exists for all $x \in \mathbb{R}$.

The CGF contains information on the moments of the distribution. In particular

$$g'(0) = E[\delta] \equiv \mu, \; g''(0) = Var[\delta] \equiv \sigma^2,$$
$$g'''(0) = \sigma^3 \cdot \text{skewness}, \; g^{(4)}(0) = \sigma^4 \cdot \text{excess kurtosis}. \tag{1}$$

The stock is in price-inelastic supply s, which is uncertain for large traders. The supply s has a full support and is independent of a terminal dividend δ.[9] I interpret the supply as being provided by a block trader who trades with a market order. In this interpretation, the uncertainty about s is due to the uncertainty about the identity of the block trader. The uncertainty of the supply is important because without it there is a dramatic multiplicity of equilibria, as is common in the literature (see, e.g., Klemperer and Meyer (1989), Vayanos (1999)). The independence of the supply s and the terminal dividend δ implies that there is no information to be learned from observing s. With uninformative supply, its distribution does not affect the equilibrium. The independence assumption is relaxed in the section 5.

The large traders are identical and maximize expected utility from their terminal wealth W. I refer to a problem solved by the traders as problem $\mathcal{P}$

$$\max_{x(p)} E_{\delta,s}[-\exp(-\gamma W)],$$

$$s.t.: W = \hat{x}\delta - \hat{p}(\cdot)\hat{x}.$$

As can be seen from the above, all traders have CARA utility with risk aversion γ. They have no initial inventories of stocks or bonds.[10] Their strategy is a bid (demand schedule) $x(p)$: the quantity of the risky asset they wish to buy ($x > 0$) or sell ($x < 0$) at a price p. Traders are *large* in the sense that they can affect market clearing prices $\hat{p}(\cdot)$ and account for this effect. I use the notation $\hat{p}(\cdot)$ (not just $\hat{p}$) to emphasize the latter fact.[11]

The trading mechanism is a uniform-price double auction. I denote the outcomes of the auction (market clearing price and allocations) by hat. The quantity $\hat{x}$ allocated to a trader is his bid evaluated at the market clearing price $\hat{p}(\cdot)$, $\hat{x} = x(\hat{p}(\cdot))$. The market clearing price $\hat{p}(\cdot)$ is determined as follows.

[9] In what follows, I will abuse notation and denote the random variable and its realization by the same letter. E.g., s may denote the random supply or its particular realization.

[10] This assumption is made to simplify the exposition. One can easily relax it as long as initial inventories are symmetric across traders.

[11] To be more rigorous, $\hat{p}(\cdot)$ is not a scalar (as would be implied by notation $\hat{p}$) but a functional that maps bids of all traders onto market clearing prices.

Given the bids of the traders $(x_i(p))_{i=\overline{1,L}}$ and a particular realization of s, the equilibrium price $\hat{p}(\cdot)$ is the one that clears the market

$$\sum_{i=1}^{L} x_i(p) = s.$$

If there are several market clearing prices, then the smallest of them is chosen. If there are no market clearing prices, then there is excess demand at all prices. The price is set to $p = \infty$ if the excess demand is positive and to $p = -\infty$ if it is negative. Such rules are the same as in Kyle (1989) and provide well-defined prices for all possible strategies of the traders. In equilibrium, the bids will be such that the finite market clearing price always exists and is unique.

Throughout the paper, I use the following notation. The *certainty equivalent of a position y in the risky asset* is denoted by $f(y)$. By definition, $f(y)$ solves

$$\exp(-\gamma f(y)) \equiv E_\delta[\exp(-\gamma y \delta)].$$

It is clear that the certainty equivalent $f(y)$ is related to the cumulant-generating function as follows

$$f(y) = -\frac{1}{\gamma} g(-\gamma y). \tag{2}$$

It can also be shown that the function $f(y)$ is strictly concave.[12]

I also introduce the *index of imperfect competition* $\nu \in (0; 1]$:

$$\nu = \frac{1}{L-2} \in (0; 1].$$

Higher values of the index correspond to a less competitive market, and the limiting case $\nu \to 0$ corresponds to perfect competition.

3 Equilibrium

The equilibrium concept is a symmetric Nash equilibrium (simply *equilibrium* in what follows) and is formally defined below.

Definition 1. A bid $x(p)$ is a symmetric Nash equilibrium if for any $i = 1, 2, ..L$, given that the traders $j \neq i$ submit bids $x_j(p) = x(p)$, it is optimal for trader i to submit bid $x_i(p) = x(p)$.

I will focus on equilibria in which the bids are continuously differentiable. Before proceeding to equilibrium characterization, I define several objects that will be used in the following parts of the paper.

[12] See, for example, Gromb and Vayanos (2002), Lemma 1. It also follows from a strict convexity of a cumulant-generating function (e.g., Billingsley (1995)).

3.1 Residual supply, ex post maximization and price impact

Given other traders' equilibrium bids $(x_j(p))_{j\neq i}$ for a given realization of s, rewrite the market clearing condition as

$$x_i(p) = s - \sum_{j\neq i} x_j(p). \tag{3}$$

The right-hand side of the equation above,

$$R_i(p;s) = s - \sum_{j\neq i} x_j(p), \tag{4}$$

is called the *residual supply*. The subscript i indicates that this is a residual supply faced by the trader i.[13] The inverse function, $P_i(x;s)$, solving

$$x = s - \sum_{j\neq i} b_j(P_i(x;s)), \tag{5}$$

is called the *inverse residual supply*.[14] The locus

$$\mathcal{C}_i(s) = \left\{ (x,p) |\, x = s - \sum_{j\neq i} x_j(p) \right\}$$

is called the *residual supply curve*.

As follows from (3), from the perspective of a particular trader, the market clearing price and quantity are determined by the intersection of his bid and the residual supply curve. Each realization of s corresponds to a horizontal parallel shift of the residual supply curve $\mathcal{C}(s)$. There is a price-quantity pair $M^*(s) = (x^*(s), p^*(s))$ on $\mathcal{C}(s)$ that maximizes the utility of a trader. I call the point $M^*(s)$ the *optimal* point. If there exists a bid that intersects with each realization of the residual supply at the optimal point, then such a bid is ex post optimal, i.e., it produces optimal price and quantities *for any* realization of s . A bid given parametrically (as a function of s) by $(x^*(s), p^*(s))$ is clearly such a bid. This bid is clearly optimal ex ante and solves problem $\mathcal{P}$.

The process of finding the equilibrium bid can therefore be simplified to finding the locus of ex post optimal points $(x^*(s), p^*(s))$ corresponding to different realizations of s. The locus $(x^*(s), p^*(s))$ is a parametric representation of the bid. The latter problem, which I call *ex post optimization* and denote $\mathcal{P}_{EP}$, can be written as[15]

$$(x^*(s), p^*(s)) = \arg\max_{x,p} \{f(x) - p\cdot x\}, \tag{6}$$
$$\text{s.t.: } (x,p) \in \mathcal{C}(s).$$

[13] Because the equilibrium is symmetric, I will often omit the subscript i for the residual supply when it does not cause confusion.

[14] The equilibrium bids are shown to be strictly decreasing. Therefore, the inverse functions in this section are well defined.

[15] The idea of finding the equilibrium in games in which players submit demand schedules by means of ex post optimization is not new. Klemperer and Meyer (1989) and Kyle (1989) were among the first to develop it.

The optimal point maximizes the certainty equivalent of a position x in risky asset $f(x)$ minus the costs of reaching this position, $p \cdot x$, on a given residual supply realization. Figure 1 provides an illustration of the ex post maximization procedure outlined above.

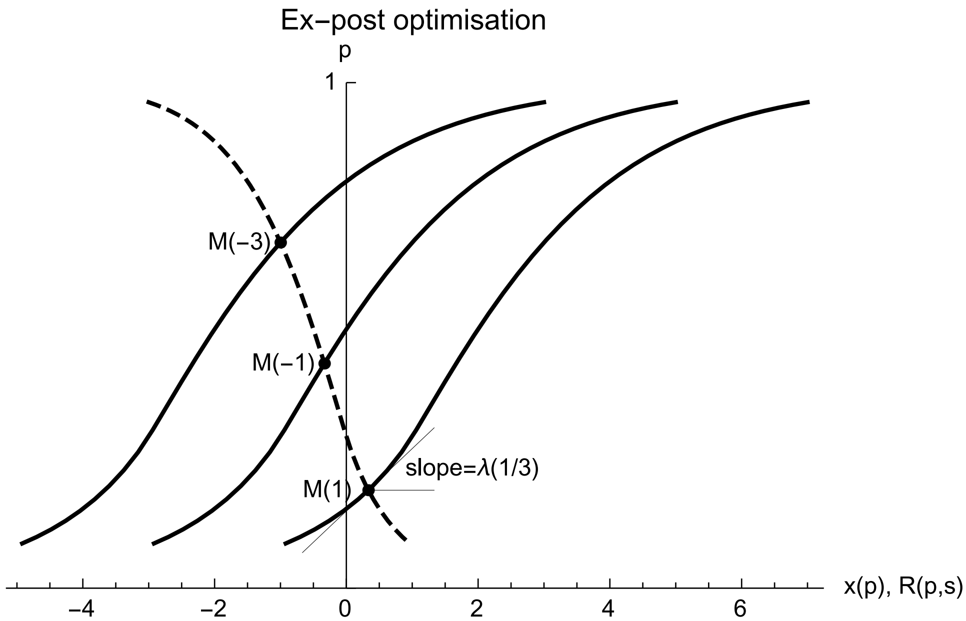

Figure 1: The figure shows realizations of the residual supply curve corresponding to the realizations of supply $s \in \{-3, -1, 1\}$ (thick black solid lines). On each curve, there is an optimal point $M(s)$ that is marked explicitly in the figure. The set of such optimal points represents the equilibrium bid (dashed line). The figure also shows the price impact at point $x = 1/3$: it is a slope of the inverse residual supply corresponding to $s = 1$ ($L = 3$ for the figure above).

Apart from being helpful in ex post maximization, the residual supply is also a useful object because it allows one to define the *price impact.* I define it analogously to Kyle's lambda: it is the slope of the equilibrium inverse residual supply evaluated at the quantity allocated to the trader in equilibrium. More precisely, consider a profile of equilibrium bids of the traders. Denote by $s_i(x^*)$ the residual supply realization such that a trader i is allocated x^* given the profile of equilibrium bids (in the symmetric equilibrium, $s_i(x^*) = Lx^*$). The price impact is by definition the slope of $P_i(x, s_i(x^*))$ evaluated at x^*

$$\lambda_i(x^*) \equiv \left.\frac{\partial P_i(x, s_i(x^*))}{\partial x}\right|_{x=x^*}. \tag{7}$$

I show that the equilibrium bids are strictly decreasing. This implies that the inverse residual supply is strictly increasing and there is always only one point at which it intersects with the equilibrium bid. Therefore, for a given x^*, there is only one realization of the inverse residual supply that intersects with the bid at x^* and only one corresponding value of the slope $\left.\frac{\partial P(x,s(x^*))}{\partial x}\right|_{x=x^*}$. Therefore, λ is a well-defined object: it is *a function* of x^* , and it *depends on* x^* *only* (and does not depend, for example, on s).

The price impact shows the equilibrium price sensitivity. For a given realization of s (producing x^* for a trader of interest), holding the bids of other traders fixed, if the trader of interest modifies his bid in a way that he is allocated $x^* + \Delta$ instead of x^* , the price will change by $\lambda(x^*) \cdot \Delta$.

3.2 Characterization of the equilibrium

I first derive the equilibrium characterization heuristically to show the intuition and then justify the derivation in the Theorem 1 below.

Consider first a competitive (price-taking) trader and his equilibrium bid. He solves

$$\max_x f(x) - px. \tag{8}$$

His inverse bid $p = I(x)$ is determined by the first-order condition in the above problem

$$f'(x) = p.$$

A strategic trader accounts for the fact that he can move prices, and his first-order condition will have a new term that is due to price impact

$$f'(x) - \frac{\partial p}{\partial x}x = p. \tag{9}$$

Once the large trader knows the price impact $\frac{\partial p}{\partial x}$, he is able to solve for his optimal bid from the first-order condition above.

Suppose that each trader has a *model* $l(x) = \frac{\partial p}{\partial x}$ of his price impact. This function shows how much, at the margin, the trader of interest can move prices if he trades x. This model, together with first-order condition (9), determines his optimal (inverse) bid

$$f'(x) - l(x)x = p. \tag{10}$$

In a Nash equilibrium, the models cannot be arbitrary. The way to pin down the equilibrium price impact model is to require it to be *consistent* with equilibrium demands of the other traders.The *consistency* condition requires assumed price impact $l(x)$ to be equal to the equilibrium one, the slope of the equilibrium inverse residual supply, i.e., $l(x) = \lambda(x)$. Intuitively, an inconsistent price sensitivity model will produce suboptimal bids and, therefore, cannot be an equilibrium because the traders will have incentives to deviate.

Consistency implies an ODE for price impact function. In the symmetric equilibrium, there are $(L-1)$ identical bids contributing to the slope of the residual supply. The slope of the residual supply is thus

$$\text{slope of the residual supply} = -(L-1)\frac{1}{I'(x)}.$$

The minus is to account for the fact that the residual supply is upward-sloping while the bid is downward-sloping; I also exploit the fact that the slope of the bid is the reciprocal of the slope of its inverse. The slope of the *inverse* residual supply is the reciprocal of the above; therefore,

$$\lambda(x) = \frac{-1}{L-1}I'(x). \tag{11}$$

The inverse bid is given by (10), which, accounting for the consistency condition $l(x) = \lambda(x)$ becomes $I(x) = f'(x) - \lambda(x)x$. The slope of the inverse bid is thus

$$I'(x) = f''(x) - \lambda'(x)x - \lambda(x).$$

Combining the above and (11) yields the ODE

$$x\lambda'(x) = (L-2)\lambda(x) + f''(x). \tag{12}$$

The equilibrium bid should therefore satisfy two conditions: the *optimality* (equation (10)) and *consistency* of the model (equation (12)). The third condition was used implicitly: for the inverse bid and the inverse residual supply to exist, the bids should be *monotone*. The analysis in Appendix C shows that monotonicity should hold in equilibrium, as the second-order conditions are violated otherwise. The theorem below summarizes and justifies the above heuristic derivation.

Theorem 1. *The bid $b(p)$ is an equilibrium if, and only if, it satisfies the following three conditions:*

(1) The bid $b(p)$ is ***optimal*** *given a model $l(x)$ of price impact*

$$f'(x) - xl(x) = p \Rightarrow x = b(p). \tag{13}$$

(2) The assumed models of price impact are ***consistent****, i.e., $l(x) = \lambda(x)$. The latter condition is equivalent to the following ODE*

$$x\lambda'(x) = (L-2)\lambda(x) + f''(x). \tag{14}$$

(3) ***Monotonicity:*** $0 < \lambda(x) < \infty \, \forall x$.

I follow Rostek and Weretka (2015) in using the representation of equilibrium through optimality and consistency conditions because it captures nicely the decision-making of real traders. As in real life, traders have a market impact model and determine optimal bids (do the optimal trade execution) given this model. In a Nash equilibrium, the price impact model is specified by the consistency condition. The main contribution relative to Rostek and Weretka (2014) is to derive the consistency condition when price impact is a function, not a constant. The fixed point condition then results in an ODE, not an algebraic equation.

The above representation is also very useful for solving the model. To find an equilibrium bid, one has to find the equilibrium price impact, which satisfies a linear ODE (14). The latter is easy to solve with standard methods.

3.3 Solution

In this section, I provide the solution to the model. I solve for the equilibrium strategies of traders, their bids, and the corresponding equilibrium price impacts. Before doing so, I introduce the new object that will be helpful in the analysis that follows.

3.3.1 Risk function

I introduce the following transformation of the cumulant-generating function, which I call the *risk function*. The risk function with a parameter a is given by

$$\rho_a(x) = \int_0^1 g''(-t^{1-a}\gamma x)dt.$$

This function summarizes the relevant risk inherent in the distribution of the terminal payoff δ when the trader changes his position in the risky asset from zero to x.

As can be seen from the above, for the normal distribution for which all risk is summarized by the variance (σ^2), the risk function is equal to σ^2. It is also clear that the risk function evaluated at zero is equal to variance

$$\rho_a(0) = g''(0) = \sigma^2.$$

However, the main justification for the function $\rho_a(x)$ as a measure of risk comes, of course, from the equilibrium. We will see that the risk function appears in the expressions for equilibrium objects in a general case where variance is present in a CARA-normal case. See the remark below.

Remark 1. We will see that the results under a general distribution can be obtained from a CARA-normal case by substituting $\rho_a(x)$ instead of σ^2 as a measure of risk. The value of parameter a differs depending on where the risk function appears. The risk contributing to price impact function (arising due to strategic interactions) is measured by the risk function with the parameter a related to the degree of competition in the economy, $a = 1 + \nu$. In all other cases, the parameter a is equal to zero. In other words, one can obtain the results under the general distribution by substituting $\rho_{1+\nu}(x)$ instead of σ^2 when computing the price impact function and substituting $\rho_0(x)$ instead of σ^2 in all other cases. The shapes of liquidity measures and the important comparative static results will be determined by the properties of risk functions that I present in Appendix B.

3.3.2 CARA-normal benchmark

I begin by investigating the CARA-normal version of my model. The results for this case are well known, and the following corollary to Theorem 1 summarizes them.

Corollary 1. *Suppose $\delta \sim N(\mu, \sigma^2)$. There exists a linear equilibrium, given by*

$$\lambda = \nu\gamma\sigma^2; \tag{15}$$

$$I(x) = \mu - \gamma\sigma^2 x - \lambda x. \tag{16}$$

Proof. In a linear equilibrium, the price impact is constant. Plugging $\lambda' = 0$ into (14) yields that $\lambda = -\nu f''(x)$. For a normal distribution, $f''(x) = -\gamma\sigma^2$, and hence (15) and (16) obtains. □

Equation (15) demonstrates that there are three sources generating the illiquidity (λ): imperfect competition (ν), limited risk-bearing capacity (γ) and the riskiness of the asset (σ^2). The higher the risk, the lower the risk-bearing capacity; the lower the competition, the higher the price impact.

Further observe that relative to the competitive case (for which $\nu = 0$), the traders reduce their bids: they bid smaller quantities for a given price. This is a consequence of imperfect competition among traders and is a standard result for divisible good auctions (see, e.g., Ausubel et al. (2014)).

3.3.3 General case

To construct an equilibrium in the general case, I divide the problem into two. I first find the equilibrium price impact. Once it is found, it is easy to find the equilibrium bid from the first-order condition (13).

According to Theorem 1, a function $\lambda(x)$ is an equilibrium price impact if and only if it satisfies the ODE (14) and $0 < \lambda(x) < \infty$. The equation (14) is easy to analyze: it is a linear ODE.

For $x \neq 0$, multiply both parts of equation (14) by the integrating factor x^{1-L} and rearrange to obtain

$$\left(x^{2-L}\lambda(x)\right)' = f''(x)x^{1-L}. \tag{17}$$

Integrating the above between x and x_0, where $x > x_0 > 0$, indicates that the solution can be written as

$$\lambda(x) = x^{L-2}\left(\lambda(x_0)x_0^{L-2} + \int_{x_0}^{x} f''(t)t^{1-L}dt\right). \tag{18}$$

An analogous equation can be written for the case in which $x < x_0 < 0$. Any solution, corresponding to different boundary conditions $\lambda(x_0)$, such that $0 < \lambda(x) < \infty$, will be an equilibrium price impact.

Two things are evident from equation (18). First, there might be many equilibrium price impact functions, as there might be many boundary conditions $\lambda(x_0)$ such that $0 < \lambda(x) < \infty$. Second, for the price impact function to exist for all real x (in particular, at infinity), some technical conditions regarding the certainty equivalent $f(x)$, or, equivalently, on CGF $g(x)$ have to be met for the integral $\int_{x_0}^{x} f''(t)t^{1-L}dt$ to converge as $x \to \infty$. Those technical conditions yield the restrictions on the CGF that need to be satisfied for the equilibrium to exist. I discuss the multiplicity and the technical conditions needed for the existence of equilibrium in a greater detail in Remarks 2 and 3. Proposition 1 below summarizes the solution.

Proposition 1. *(1) The equilibrium exists if and only if $\rho_{1+\nu}(x) < \infty$. In particular, it exists for any distribution with bounded support.*

(2) The equilibrium price impact function is given by

$$\lambda(x) = \nu\gamma\rho_{1+\nu}(x) + \mathbb{I}(x \geq 0)C^{+}x^{L-2} + \mathbb{I}(x < 0)C^{-}\cdot(-x)^{L-2},$$

for any $C^{+}, C^{-} \geq 0$.

(3) The equilibrium inverse bid is given by

$$I(x) = \mu - \gamma\rho_0(x)x - \lambda(x)x. \tag{19}$$

I comment on the technical conditions that need to be satisfied for the equilibrium to exist and on the multiplicity in the two remarks below.

Remark 2. According to Proposition 1, the equilibrium exists if and only if the risk function $\rho_{1+\nu}(x)$ is finite for every finite x. After some algebra, the latter function can be written as

$$\rho_{1+\nu}(x) = (L-2)\int_{1}^{\infty} g''(-\gamma yx)y^{1-L}dy.$$

For the integral to converge, the second derivative of the CGF should grow not too fast as $y \to \pm\infty$. An example of the distribution for which the equilibrium does not exist is a Poisson distribution for which the CGF and its second derivative grow exponentially.

However, given that equilibria exist for any distribution with bounded support, I believe that the technical conditions are not too restrictive. Indeed, in the real world, the payoff cannot be unbounded. There is a lower bound, which is due to limited liability, and there is an upper bound, which is due to the fact that the resources of are limited (and hence an asset cannot have an infinite payoff). It should also be noted that the technical conditions hold also for distributions with unbounded support: examples include (and are not restricted to) normal (the benchmark) and the mixture of normal distributions.

Remark 3. Despite the presence of uncertainty regarding the supply, there is nevertheless a multiplicity of equilibria. The mechanism behind the multiplicity is the following.

Technically, according to Theorem 1, the equilibrium lambda is a price impact model that is consistent. The consistency condition is equivalent to ODE (14). The standard result from the theory of differential equations implies that having obtained a solution to a linear ODE (14) another can be obtained by adding a solution of an homogenous ODE, in this case $x\lambda'(x) = (L-2)\lambda(x)$. The latter solution is given by $\mathtt{1}(x \geq 0)C^+x^{L-2} + \mathtt{1}(x < 0)C^-\left(-x\right)^{L-2}$. After filtering out solutions not satisfying condition (3) of Theorem 1, we are left with $C^+, C^- \geq 0$.

In economic terms, the multiplicity can be attributed to the following complementarity: if a trader believes that the price impact is high, he provides less liquidity, which implies a higher price impact for other traders. This, in turn, induces them to provide less liquidity, confirming the higher price impact for a trader of interest.

Note that the multiplicity is present even in the standard CARA-normal case. The multiplicity result in a CARA-normal case is known in the divisible goods auctions literature (see Wang and Zender (2002), Proposition 3.4). The non-uniqueness of equilibrium arises for a similar reason in Bhattacharya and Spiegel (1991). However, in the generalized setting of my paper a selection argument, exploiting the generality of distribution is possible.

In Glebkin, Rostek and Yoon (2015) we study the uniqueness of equilibrium in supply functions. Parts (1) and (2) of the following Proposition is from there and are included here for completeness. The Proposition demonstrates that there is only one equilibrium satisfying the following natural properties.

Proposition 2. *The equilibrium with $C^+ = C^- = 0$ is the unique equilibrium having the following properties:*

(1) Suppose that δ has a bounded support (a, b). The equilibrium with $C^+ = C^- = 0$ is the unique equilibrium with equilibrium prices within $[a, b]$.

(2) Suppose that δ has infinite support. Consider an asset with a payoff $\delta_n = \delta \cdot \mathbb{I}(\delta \in (a_n, b_n))$, where a_n and b_n are finite. Denote $p_n(s)$ the price of the asset that pays δ_n when the supply realization is s in the unique equilibrium in which $p_n(s)$ is within (a_n, b_n) for all s. Then $p_n(s)$ converges pointwise to $p(s)$ as $a_n \to -\infty$ and $b_n \to \infty$, where $p(s)$ is a price of the asset that pays δ in the equilibrium with $C^+ = C^- = 0$.

(3) The equilibrium with $C^+ = C^- = 0$ is the unique equilibrium in which $\lambda(s)$ converges to zero pointwise as either $\nu \to 0$ or $\gamma \to 0$.

In the CARA-normal model, the equilibrium with $C^+ = C^- = 0$ *corresponds to the linear equilibrium of corollary 1.*

The first property is intuitive: if the price is outside payoff bounds the block trader gets negative profit with certainty and hence should not trade. Block trader is unmodeled and is assumed to submit a price-inelastic bid: he is willing to pay any price. However, the latter is only consistent with rational behavior if prices outside the bounds of payoff never realize in equilibrium. Another piece of intuition is as follows. By assumption, there is a full support uncertainty regarding the supply, i.e., any quantity may be supplied with positive probability. Suppose, for example, that for some supply realization s the price is outside the payoff bounds. Then, the trader providing the supply will realize a negative profit with certainty, for any realization of δ. However, it is then not consistent to assume that quantity s may be supplied with positive probability.

The second property extends the first one to the case of infinite support. It highlights that price of an asset with arbitrary close, but bounded payoffs (for which the price can be uniquely pinned down requiring the property (1) to hold) is close to the price of an asset with unbounded payoff in the equilibrium with $C^+ = C^- = 0$.

The third property highlights that price impact vanishes as the sources of illiquidity disappear (the economy becomes perfectly competitive, $\nu \to 0$; or risk-bearing capacity becomes infinite, $\gamma \to 0$) only in the equilibrium with $C^+ = C^- = 0$.

Henceforth, I focus on the equilibrium with $C^+ = C^- = 0$ (and will simply refer to it as *equilibrium* hereafter). Although other equilibria may be of theoretical interest, they appear to have little empirical relevance. Indeed, unbounded prices are not observed for assets with bounded payoffs (e.g., negative prices on limited liability assets), and there is empirical evidence that competition reduces illiquidity (e.g., Kagel and Levin (2001)).

4 The shape of the price impact

The purpose of this section is to study how the prices in the imperfectly competitive market may be affected by a block sell or buy order s. This is measured by several liquidity measures that I describe below.

I will use the notation $\bar{s}$ for per capita supply

$$\bar{s} \equiv s/L.$$

A positive $\bar{s}$ corresponds to a sell order, whereas a negative $\bar{s}$ corresponds to a buy order.

Denote $p(s)$ as the equilibrium price when the supply is s. The function $p(s)$ is easy to find. Because each large trader is allocated s/L in the symmetric equilibrium, the equilibrium price can be determined from condition (13), which using the risk function, can be written as (cf. (19))

$$p(s) = \mu - \gamma\rho_0(\bar{s}) - \lambda(\bar{s})\bar{s}. \tag{20}$$

The price reaction function $\Pi(s) = |p(s) - \mu|$ is the absolute value of the difference between the equilibrium price when the supply is s and the equilibrium price when the supply is zero (which is equal to μ). It measures the total reaction of the equilibrium price to a block of size s. Equation (20) yields

$$\Pi(s) = \gamma\rho_0(\bar{s})|\bar{s}| + \lambda(\bar{s})|\bar{s}|. \tag{21}$$

The price reaction function can be decomposed into two parts. The first

$$\tau(\bar{s}) \equiv \lambda(\bar{s})|\bar{s}| = \nu\gamma\rho_{1+\nu}(\bar{s})|\bar{s}|, \tag{22}$$

arises due to strategic interactions. I will refer to it as a *strategic component* of price reaction function. It is (the absolute value of) the difference between price $p(s)$ and price $f'(\bar{s})$ that a competitive trader would pay.[16] In the limit, $\nu \to 0$, $s/L = const = \bar{s}$, corresponding to perfect competition, this strategic component disappears.

The second component

$$\pi(s) \equiv \gamma\rho_0(\bar{s})|\bar{s}|, \tag{23}$$

Is non-strategic. It is the equal to $|f'(s/L) - \mu|$, the price reaction in a perfectly competitive economy. I will refer to it as a *non-strategic component* of price reaction function.

In a dynamic CARA-normal model, Rostek and Weretka (2014) demonstrate that the component of the price reaction due to strategic interactions is transitory. This is intuitive: the strategic component in a particular period (say t) arises because the traders account for their price impact in this particular period. The extent to which a trader can move prices in period t does not matter for him in periods $t+1$ and so on because the price at t is already realized. Therefore, there will be a price effect at time t and no effect at times $t+1$ and so forth. They also show that the non-strategic component, in contrast, is permanent. This is also intuitive. If a block trader sells in period t, he increases the inventory of traders who absorb this trade. They have greater inventories in periods t,$t+1$, .. and, being risk averse, require a price discount in those periods. The price effect lasts for subsequent periods and is thus permanent.

Given the above, one can interpret the strategic component $\tau(\bar{s})$ as a temporary price effect and the non-strategic component $\pi(\bar{s})$ as a permanent price effect. See also the remark below.

Remark 4. One can also justify the difference between the temporary and the permanent price effects as follows. I allow traders to trade in two additional periods, $t = -1/2$ and $t = 1/2$. However, given the complexity of a dynamic problem without normality, I assume that in each period traders behave myopically and do not foresee the possibility to trade in the future. There is a block sale s at $t = 0$, and the supply at $t = -1/2$, $1/2$ is zero. As before, the traders consume only at time $t = 1$.

At time $t = -1/2$, there will be no trading, as traders are symmetric and there is no supply provided. The price at which the traders having no initial endowments will neither buy nor sell is

$$p_{-1/2} = \mu.$$

The price at $t = 0$ is already found and is given by

$$p_0 = \mu - \gamma\rho_0(\bar{s})\bar{s} - \lambda(\bar{s})\bar{s}.$$

[16]It is easy to show that $f'(\bar{s}) = \mu - \gamma\rho_0(\bar{s})$.

At time $t = 1/2$, there will also be no trading due to symmetry and the absence of supply. However, each trader now has an endowment $\bar{s}$; therefore, the price at which traders will be indifferent between buying and selling is

$$p_{1/2} = f'(\bar{s}) = \mu - \gamma\rho_0(\bar{s})\bar{s}.$$

The immediate price reaction to block selling is thus $p_0 - p_{-1/2} = \gamma\rho_0(\bar{s})\bar{s} + \lambda(\bar{s})\bar{s}$; however, in the longer term, the strategic component disappears, $p_{-1/2} - p_{1/2} = \gamma\rho_0(\bar{s})\bar{s}$. The strategic component is therefore temporary, while the non-strategic component is permanent.

4.1 Theoretical results

In this section, I investigate the shapes of liquidity measures (price reaction function, and its' two components), focusing on the monotonicity, convexity and asymmetry of those functions. These properties have been the focus of empirical studies.

The expressions for the price reaction function and its' components are given by (21), (22) and (23), respectively. It then follows that for a normal distribution, the three functions are linear in the absolute value of the order size $|\bar{s}|$, and are also symmetric:

$$\Pi(s) = \gamma(1+\nu)\sigma^2 \cdot |\bar{s}|,\ \tau(s) = \gamma\nu\sigma^2 \cdot |\bar{s}| \text{ and } \pi(s) = \gamma\sigma^2 \cdot |\bar{s}|. \tag{24}$$

The analytical results in the general case can be obtained in two limits: when the order size is small ($|\bar{s}| \to 0$) and when it is large ($|\bar{s}| \to \infty$). Those two limits are associated with the two forces that play no role under normality. Higher moments play a role in the $|\bar{s}| \to 0$ limit, whereas the bounds of the support are important in the $|\bar{s}| \to \infty$ limit. The two limits, and the two respective forces, are examined separately below.

Proposition 3. *The price reaction function and its' non-strategic component are strictly increasing in the absolute value of the order size $|\bar{s}|$. Suppose that δ has bounded support $\delta \in [a, b]$. The following is true in the $|\bar{s}| \to \infty$ limit:*

buy order	sell order
$\Pi(-\infty) = b - \mu$	$\Pi(+\infty) = \mu - a$
$\pi(-\infty) = b - \mu$	$\pi(+\infty) = \mu - a$
$\tau(-\infty) = 0$	$\tau(+\infty) = 0$

Consequently, there exists $x_1 > 0$ such that the following is true for $s : |s| > x_1$:

Liquidity measure $L(s)$	$\text{sign}(L(s) - L(-s))$
$\Pi(s)$	$\text{sign}((\mu - a) - (b - \mu))$
$\pi(s)$	$\text{sign}((\mu - a) - (b - \mu))$

If $g'''(x)$ does not change sign given a large enough x, the price reaction function $\Pi(x)$ is concave for a large enough x.

It is intuitive that the price reaction function and the its' non-strategic component are monotone: larger orders have larger price effects. Finite limits at infinity and monotonicity imply that they have a horizontal asymptote $b - \mu$ $(\mu - a)$ at minus (plus) infinity. This is also intuitive: the price reaction

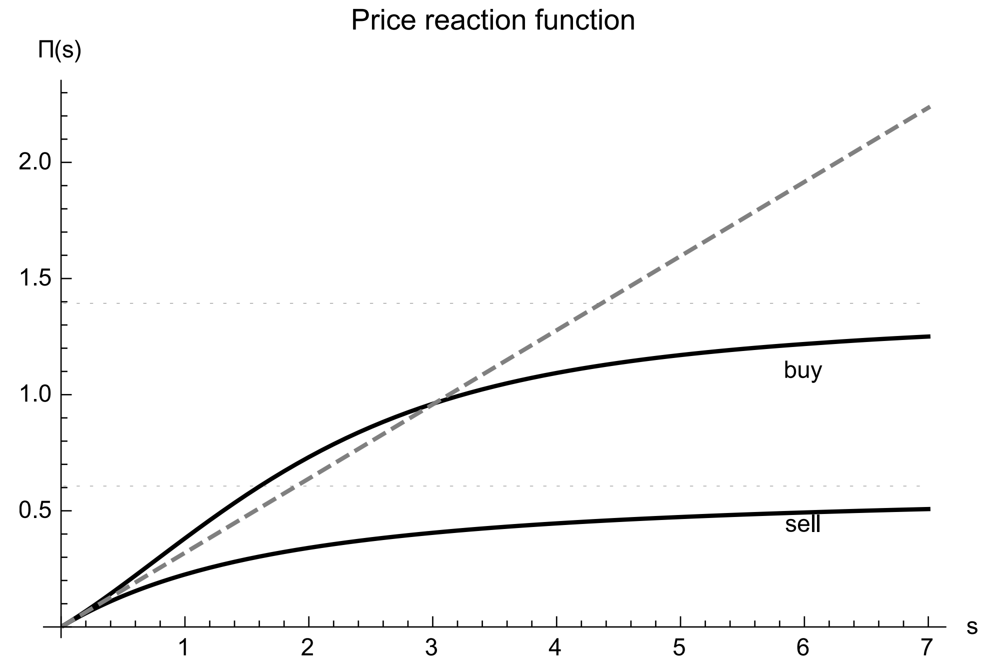

Figure 2: If the payoff is bounded, and hence the price of the asset and the price reaction function should also be bounded. The figure represents the price reaction function for the case of a normal distribution $N(0,1)$ truncated to the segment$[0,2]$ (solid lines with buy and sell orders labeled accordingly). It compares the latter to the price reaction function for the case of a normal distribution (with mean 0.72 and variance 0.25 equal to that of the truncated normal distribution) represented by the dashed line. For a normal distribution, the function is linear, unbounded and the same for buy and sell orders. For the truncated normal distribution, the function is asymmetric and is bounded by the horizontal dotted lines. The combination of boundedness and monotonicity produces the concave-looking shape. The shape is not exactly concave: the price reaction function for buy orders is above the tangent line (and hence convex) in the neighborhood of zero, consistent with Proposition 4.

$\Pi(\cdot)$ cannot be higher than $b-\mu$ $(\mu-a)$ for a very large buy (sell) order because otherwise the price itself would be higher than b (lower than a) and the block trader should not trade. The non-strategic part $\pi(\cdot)$ is a price reaction in a competitive economy; therefore, the latter argument also applies to it.

If the support is unbounded, the price reaction and its' non-strategic component would also be unbounded. The bounds of the distribution represent a force that bends those functions preventing violation of the bounds. The functions therefore cannot be convex. Moreover, a concave shape is typically observed. This is illustrated in figure 2, where I compare the price reaction function in the normal and truncated normal cases. Under the condition that the third derivative of the CGF does not change sign given a large enough x, it is possible to prove that the price reaction function is concave for large order sizes.[17]

The strategic component has to decrease to zero at infinity. This is because $\Pi(s)=\pi(s)+\tau(s)$ and the limits of $\Pi(s)$ and $\pi(s)$ coincide. Intuitively, the non-strategic component already drives prices toward the bounds of the payoff's support; therefore, if the non-strategic component is not zero, the price bounds will be violated. Therefore, the force bending the price reaction function and the non-strategic

[17]I could not find an example of a distribution with bounded support for which this condition does not hold. I believe that this condition is not too restrictive.

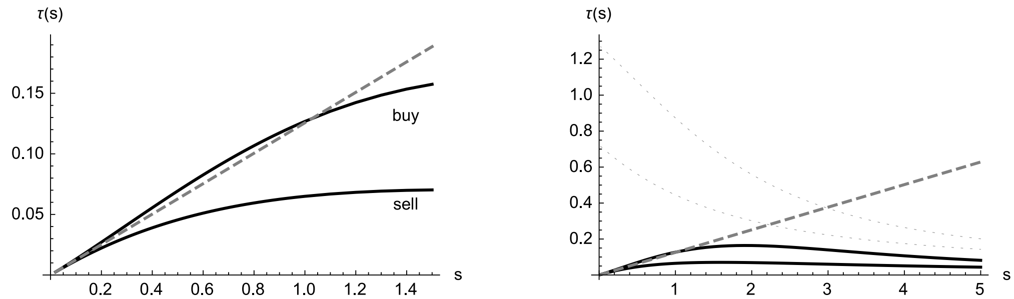

Figure 3: The bounds of the payoff produce a strategic component of price impact with a concave-alike shape if the order size is not very large (left panel). However, the force that shaped the price reaction function is even stronger for its' strategic component: it is bounded not by a horizontal line but by $\mu - a - \pi(s)\,(b - \mu - \pi(s))$ for sell (buy) orders represented by a dotted line (right panel). The plots are for a normal distribution $N(0,1)$ truncated to the segment$[0,2]$

component is even stronger, making the strategic component non-monotone. Due to the presence of this force, the function $\tau(s)$ cannot be convex. If attention is restricted to the monotone part of $\tau(s)$, its shape also appears concave, as illustrated by the left pahnel of Figure 3.

The asymmetry of the price reaction function at infinity obtains if $\mu \neq \frac{b-a}{2}$, i.e., if the mean of the distribution does not coincide with the middle of its support. Intuitively, because the shape of the price impact is linked to the shape of the distribution, asymmetry in the one leads to the asymmetry in the other.

In the proposition below I investigate the shape of liquidity measures for small order size.

Proposition 4. *Suppose that $L > 3$. Denote the skewness by κ_3. There exists $x_0 > 0$ such that the following is true :*

1) Local convexity and monotonicity:

Liquidity measure $L(s)$	sign$(L'(\lvert s\rvert))$	sign$(L''(\lvert s\rvert))$
$\Pi(s)$	$+^*$	-sign$(\kappa_3 \cdot s)$
$\pi(s)$	$+^*$	-sign$(\kappa_3 \cdot s)$
$\tau(s)$	$+$	-sign$(\kappa_3 \cdot s)$

for all y if marked by $*$, and for $s : |s| < x_0$ otherwise.

2) Local asymmetry:

$$\text{sign}(\Pi(s) - \Pi(-s)) = -\text{sign}(\kappa_3)\,\forall s : \, |s| < x_0,$$

$$\text{sign}(\pi(s) - \pi(-s)) = -\text{sign}(\kappa_3)\,\forall s : \, |s| < x_0.$$

3) *Comparative statics:* $|\Pi(s) - \Pi(-s)|$ and $|\Pi''(s)|$ are increasing in ν, γ and $|\kappa_3|$ for $s : \, |s| < x_0$.

When asset payoffs are positively skewed, the price reaction function is a convex (concave) function of order size for small buy (sell) orders. Similarly, when asset payoffs are negatively skewed, the price reaction function is a concave (convex) function of order size for small buy (sell) orders.

Consider the case of a sell order and positive skewness. To understand the intuition, consider a benchmark economy in which higher moments play no role. It is identical to the initial economy, except that the asset's payoff is normally distributed with mean and variance equal to that in the initial economy. In the benchmark economy, the price reaction function is linear. This linear function is represented by a dashed line in the figure 3. Moreover, for a very small (infinitesimal) order size, the role of higher moments is negligible; therefore, the price impact function in the benchmark economy should be arbitrarily close to that in the initial economy in the neighborhood of zero. Consequently, the line representing the price impact in the benchmark economy is tangent to the price reaction in the initial economy (which the left panel of figure 3 illustrates). A concave function lies below its tangent line, and hence the remaining question is why the price impact in the benchmark economy is greater. The intuition is simple: with a positively skewed payoff, the trading profit of the investors accommodating the sale order is also positively skewed (i.e., they occasionally receive large positive surprises to their profits). Consequently, they require less price compensation relative to the case of zero skewness. Similarly, for small purchases, the shape of the price reation function is convex with positive skewness.

The above discussion also implies that with non-zero skewness the price reaction function is asymmetric for small orders. As noted above, with positive skewness the price reaction function for purchases is convex, and therefore lies above its' tangent line. It is concave for sell orders, and therefore lies below the tangent line and, consequently, below the price reaction function for purchases. Hence, with positive skewness the price reacts stronger to small purchases rather than sells. Similarly, with negative skewness the price reacts stronger to sells. Left panel of the figure 3 provides an illustration for the case of positive skewness.

4.2 An example

In this section, I examine numerically the case of a δ distributed according to the mixture of normal distributions. This example illustrates that, in theory, the shapes of the price impact can be quite rich. This case has at least two natural interpretations.

The first interpretation is as follows. Suppose that δ is a claim to cash flows generated by a firm. Suppose also that between time 0 and time 1, there is a corporate event (occurring with probability p) that may increase or decrease δ, also making it more or less risky. Suppose for simplicity that conditional on the outcome of this event, the distribution of δ is normal. The resulting distribution is then the mixture of normals

$$F_\delta(x) = p \cdot \Phi(x|\mu_1, \sigma_1) + (1-p) \cdot \Phi(x|\mu_2, \sigma_2), \tag{25}$$

where F_δ denotes the CDF of the dividend δ and $\Phi(x|\mu_i, \sigma_i)$ denotes the CDF of a normal distribution with mean μ_i and variance σ_i^2. The event can, for example, be:

- default, in which case p is the probability of default, μ_1 is a mean repayment in the event of default and $\sigma_1 \to 0$ if this repayment is certain.
- appointment of a new CEO, in which case p is the probability that a search for a new CEO is successful. If the new is CEO is better than the old one, one may expect $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$.

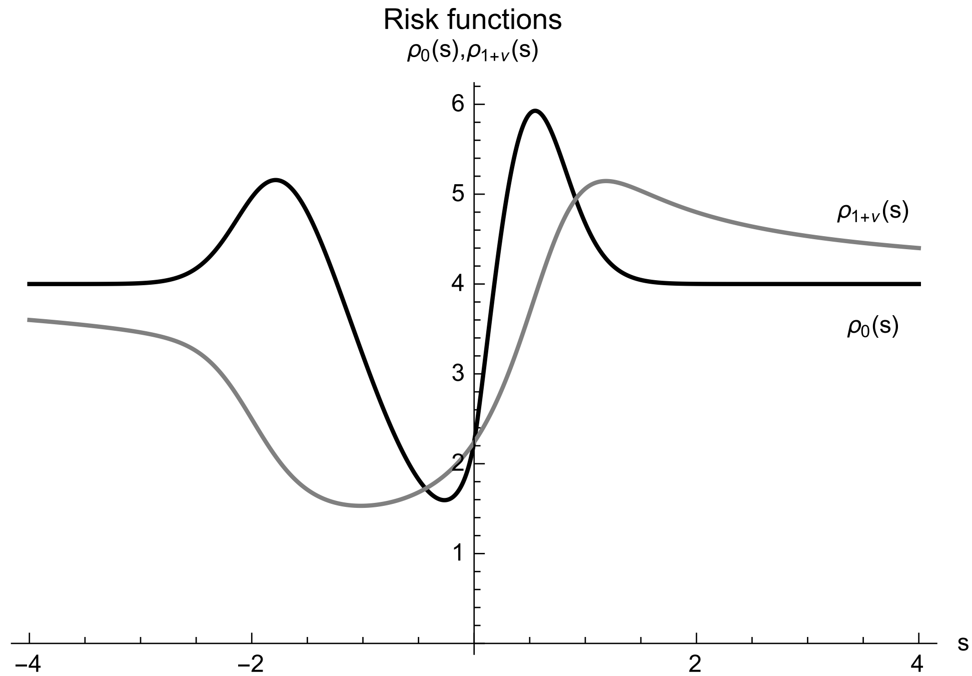

Figure 4: Mixture of normals $N(0,1)$ with probability 0.8 and $N(-2,2)$ with probability 0.2. For small order sizes, the normal with the lower variance of 1 dominates and the risk is close to 1. For large order sizes, the normal with the higher variance of 4 dominates and the risk is close to 4. In between, the risk spikes above 4 due to uncertainty regarding the variance.

The second interpretation is as follows. Suppose that all traders have a prior belief $\delta \sim N(\mu, \sigma)$ and receive the same signal ι concerning the asset. There is, however, uncertainty with respect to whether the signal is informative, in the spirit of Banerjee and Green (2014). For example, the signal might be

$$\iota = \begin{cases} \delta + \epsilon, & \text{with probability } p, \\ \epsilon, & \text{with probability } 1-p, \end{cases}$$

where δ and $\epsilon \sim N\left(0, \sigma_\epsilon^2\right)$ are independent. The posterior distribution will then be a mixture of normals (25) with $\mu_2 = \mu$, $\sigma_2^2 = \sigma^2$ and $\mu_1 = \mu + \beta(\iota - \mu)$, $\sigma_1^2 = \sigma^2 - \beta^2(\sigma^2 + \sigma_\epsilon^2)$, and $\beta = \frac{\sigma^2}{\sigma^2 + \sigma_\epsilon^2}$.

Figure 4 represents the risk functions $\rho_{1+\nu}(x)$ and $\rho_0(x)$. Recall that $\rho_{1+\nu}(|\bar{s}|)$ and $\rho_0(|\bar{s}|)$ are proportional to the per unit strategic and non-strategic components of price reaction function: e.g., for $s > 0$ we have $\tau(s)/\bar{s} = \nu\gamma\rho_{1+\nu}(\bar{s})$ and $\pi(s)/\bar{s} = \gamma\rho_0(\bar{s})$[18].

The intuition behind the shapes represented in figure 4 is the following. When the order size is small, a normal distribution with lower variance (denote it σ_1^2) dominates. The risk is thus small and is close to σ_1^2. For a large-sized order, the normal distribution with higher variance (denote it σ_2^2) dominates and the risk is close to σ_2^2. In between, the uncertainty regarding the variance increases risk and there are spikes in the risk functions. Therefore, if one aims to execute a small order, he or she should be optimistic about risk and assume that the level of risk is close to σ_1^2. If the order is very large, one

[18] The general formula is $\tau(|s|)/|\bar{s}| = \nu\gamma\rho_{1+\nu}(|\bar{s}|)$ and $\pi(|s|)/|\bar{s}| = \gamma\rho_0(|\bar{s}|)$.

should be conservative about risk and assume that the risk is close to σ_2^2. If the order is neither very large nor very small, one should be pessimistic, as the uncertainty makes the risk higher than σ_2^2.

5 Informed block trader

In this section, I relax the assumption that the block seller is uninformed, i.e., that s and δ are independent. The conditional distribution of δ given s is characterized by the conditional CGF

$$g(y, s) = \ln E[\exp(y\delta)|s].$$

As before, I assume that the distribution of s has full support. The conditional CGF and the conditional certainty equivalent, defined as $f(y, s)$, that solves

$$\exp(-\gamma f(y, s)) \equiv E[\exp(-\gamma y\delta)|s],$$

are related as follows

$$f(y, s) = -\frac{1}{\gamma} g(-\gamma y, s).$$

As before, my focus is liquidity: I will investigate how price reaction function and its' components depend on the size of supply s. I begin by deriving heuristically the equilibrium representation.

The approach to solving the problem is still ex post optimization, but now the residual supply curve realization reveals s and thus provides information on the terminal payoff. The ex post maximization problem is then written as

$$(x^*(s), p^*(s)) = \arg\max_{x,p} \{f(x, s) - p \cdot x\},$$

$$\text{s.t.: } (x, p) \in \mathcal{C}(s).$$

The first-order condition in the ex post maximization problem is given by

$$f_1(x, s) - l(x)x = p,$$

where $l(x) = \frac{\partial p}{\partial x}$ is the price sensitivity model. The above first-order condition makes it possible to determine the optimal x for a given s. In the symmetric equilibrium, $x = s/L$ should be optimal. Equivalently, x is optimal when $s = Lx$ is realized. The above can thus be rewritten as follows

$$f_1(x, Lx) - l(x)x = p = I(x). \tag{26}$$

As before, the way to specify the price sensitivity model is to require it to be *consistent*: the assumed price sensitivity should be equal to the equilibrium sensitivity. Consistency is equivalent to

$$l(x) = \lambda(x) = -\frac{1}{L-1} I'(x).$$

Combining the above and (26) yields the ODE for price impact function

$$x\lambda'(x) = (L-2)\lambda(x) + h(x),$$

where

$$h(x) \equiv \frac{d}{dx} f_1(x, Lx) = f_{11}(x, Lx) + L f_{12}(x, Lx) = -\gamma g_{11}(-\gamma x, Lx) + L g_{12}(-\gamma x, Lx). \tag{27}$$

To proceed further, I impose the following assumption.

Assumption 1. $h(x) < 0$.

The above assumption states that the supply distribution is such that the equilibrium marginal certainty equivalent $f_1(y|Ly)$ is decreasing in the equilibrium quantity y. Given that $g_{11}(x|y) > 0$ Assumption 1 is satisfied if a stronger assumption holds[19]

Assumption 1s. $f_{12}(x|s) < 0 \; \forall x, s$.

Assumption 1s says that higher supply realization is bad news for large traders. This is intuitive: supply is high when the informed traders sell a great deal, which is the case when they receive bad news about the asset.

The following theorem generalizes Theorem 1 for the setting with informed block traders.

Theorem 2. *If Assumption 1 holds, bid $b(p)$ is an equilibrium if, and only if, it satisfies the following three conditions:*

*(1) Bid $b(p)$ is **optimal** given a model $l(x)$ of price sensitivity*

$$f'(x, Lx) - xl(x) = p \Rightarrow x = b(p). \tag{28}$$

*(2) The assumed models of price sensitivity are **consistent**, i.e., the assumed models are equal to equilibrium price sensitivity, $l(x) = \lambda(x)$. The latter condition is equivalent to the following ODE*

$$x\lambda'(x) = (L-2)\lambda(x) + h(x). \tag{29}$$

*(3) **Monotonicity**: $0 < \lambda(x) < \infty \; \forall x$.*

The analysis in the main part of the paper can be generalized to the case of asymmetric information by substituting $h(x)$ for $f''(x)$. I define the *augmented risk function* $\varphi_a(x)$, which generalizes the risk function in the baseline model, as follows

$$\begin{aligned}
\varphi_a(x) &\equiv -1/\gamma \int_0^1 h(t^{1-a}x)dt = \\
&= -1/\gamma \int_0^1 \left(f_{11}(t^{1-a}x, Lt^{1-a}x) + L f_{12}(t^{1-a}x, Lt^{1-a}x) \right) dt \\
&= \underbrace{\int_0^1 g_{11}(-t^{1-a}\gamma x, Lt^{1-a}x)dt}_{\text{risk function } \rho_a(x)} \underbrace{-L/\gamma \int_0^1 g_{12}(-t^{1-a}\gamma x, Lt^{1-a}x)dt}_{\text{information function } \iota_a(x)} \\
&= \rho_a(x) + \iota_a(x).
\end{aligned}$$

[19]The conditional CGF $g(x|y)$ can be proven to be convex in x analogously to the case of the unconditional CGF, using Cauchy-Schwartz inequality.

The augmented risk function is defined analogously to the risk function in the baseline setting with the sensitivity of marginal utility $f''(x)$ substituted by $h(x)$. Because the sensitivity of the marginal utility has two components, the sensitivity to information and the sensitivity to the quantity (due to risk aversion), the augmented risk function can be decomposed into two parts: inventory risk (simply risk hereafter) and information.

The risk function is familiar: it measures the sensitivity of marginal certainty equivalent $f_1(x, s)$ to the size of the position in the risky asset x. The latter sensitivity is not zero because the asset is risky and the traders are risk averse.

The information function measures the sensitivity of marginal certainty equivalent $f_1(x, s)$ to the news provided by the supply s. A higher equilibrium quantity x of the risky asset indicates higher supply provided by the informed traders, which, in turn indicates bad news about the terminal payoff δ under Assumption 1s. Therefore, the information function summarizes the *winner's curse* faced by the large traders: if they are allocated a great deal of supply in equilibrium, this means that the suppliers sold a great deal, which indicates that the asset is of poor quality.

It is instructive to consider the following example.

Example 1. Suppose that δ and s are jointly normal. Then, by projection theorem

$$g(x, s) = (\bar{x} + \beta(s - E[s]))x + 1/2\sigma^2_{\delta|s}x^2,$$

where $\beta = \frac{cov(x,y)}{var(y)}$, and $\sigma^2_{\delta|s}$ is a conditional variance of δ given s. The risk function and the information function are given by

$$\rho_a(x) = \sigma^2_{\delta|s},\ \iota_a(x) = \beta.$$

The above indicates that in the CARA-normal case, both components of the risk function are constants. The risk function is a conditional volatility. The information function is the sensitivity of the conditional mean to the signal.

The analysis in the main part of the paper can be generalized to the case considered in this section by substituting the augmented risk function for the risk function. Following the steps of Proposition 1, one obtains the results for the case considered here. It will be clear from Proposition 6 that the solution with $C^+ = C^- = 0$ is the unique one satisfying prices being in $[a, b]$ for the bounded payoff $\delta \in [a, b]$, thus providing grounds for focusing on this equilibrium here.

As in the augmented risk function, all liquidity measures now separate into two components that are denoted by superscripts ρ (risk component) and ι (information component).

Proposition 5. *Suppose that Assumption 1s holds.*

(1) The equilibrium with asymmetric information exists if and only if $\varphi_{1+\nu}(x) < \infty$.

(2) The equilibrium price impact function is decomposed into risk and information components and is given by

$$\lambda(x) = \nu\gamma\varphi_{1+\nu}(x) = \nu\gamma\rho_{1+\nu}(x) + \nu\gamma\iota_{1+\nu}(x),$$

(3) The equilibrium price is given by

$$p = \mu - \gamma\varphi_0(\bar{s})\bar{s} - \lambda(\bar{s})\bar{s}. \tag{30}$$

The strategic and non-strategic components of price reaction function are decomposed into risk and information parts

$$\pi(\mathrm{s}) = \gamma\phi_0(\bar{\mathrm{s}})\bar{\mathrm{s}} = \gamma\rho_0(\bar{\mathrm{s}})|\bar{\mathrm{s}}| + \gamma|\iota_0(\bar{\mathrm{s}})\bar{\mathrm{s}}| \equiv \pi^{\rho}(\mathrm{s}) + \pi^{\iota}(\mathrm{s}),$$

$$\tau(\mathrm{s}) = \nu\gamma\phi_{1+\nu}(\bar{s})\bar{s} = \nu\gamma\rho_{1+\nu}(\bar{s})|\bar{s}| + \nu\gamma|\iota_{1+\nu}(\bar{s})\bar{s}| \equiv \tau^{\rho}(s) + \tau^{\iota}(s).$$

The results of the main part of the paper are nicely generalized: one has to augment the risk function with the information function to obtain equilibrium objects in the informed block trader case. The price reaction function is nicely decomposed into two components. The risk component is due to the risk aversion of the traders, and the information component is due to the winner's curse. The non-strategic component arises because traders account for their own risk aversion and the winner's curse to which they are exposed. The strategic component arises because a trader strategically recognizes that others are risk averse and exposed to the winner's curse, which affects his price sensitivity.

Below, I investigate the shape of the price reaction function in two limiting cases: $s \to \infty$ and $s \to 0$. I first examine the $s \to \infty$ limit and introduce the following assumption.

Assumption 2. The support of the conditional distribution of δ given s is the same as the support of the unconditional distribution of δ.

The above assumption states that the traders cannot improve their knowledge about the support of the terminal payoff by observing the supply. This assumption holds, for example, when the supply can be decomposed into information and noise, $s = f(\delta) + \epsilon$ and the noise ϵ has a full support[20].

Proposition 6. *Suppose that δ has a bounded support and that Assumption 1s holds. Then the price reaction function and its' non-strategic component are strictly increasing in the absolute value of the order size $|\bar{s}|$. Suppose that δ has bounded support $\delta \in [a, b]$ and Assumption 2 holds . The following is true in the $|\bar{s}| \to \infty$ limit:*

buy order	*sell order*
$\Pi(-\infty) = b - \mu$	$\Pi(+\infty) = \mu - a$
$\pi(-\infty) = b - \mu$	$\pi(+\infty) = \mu - a$
$\tau(-\infty) = 0$	$\tau(+\infty) = 0$

The proposition above confirms the robustness of the results of Proposition 3 for the case in which the block trader may be informed. As long as the available information does not allow to improve one's knowledge of the support of the distribution of δ, the boundedness of δ works in the the same way as in the uninformed case. It represents a force that bends down functions $\Pi(\cdot)$ and $\tau(\cdot)$. The functions therefore cannot be convex - exactly as in the baseline setting.

Below, I investigate the shape of the price impact in the $s \to 0$ limit. Before doing so I introduce the following notation. The conditional mean, variance and skewness of δ given s are all functions of s and are denoted by $\mu(s)$, $\sigma^2(s)$ and $\kappa_3(s)$, respectively.

[20] Indeed, $f_{\delta|s} = \frac{f_{s|\delta} \cdot f_\delta}{f_s}$, and $f_{s|\delta}$ and f_s have full support.

Proposition 7. *Suppose that $L > 3$ and Assumption 1s holds. There exists $x_0 > 0$ such that the following is true.*

1) Local convexity and monotonicity:

Liquidity measure $L(s)$	$\text{sign}(L'(\|s\|))$	$\text{sign}(L''(\|s\|))$
$\Pi^\rho(s)$	$+$	$\text{sign}\left(\left(\left(\sigma^2\right)'(0) - \gamma/L \cdot \kappa_3(0)\right) \cdot s\right)$
$\Pi^\iota(s)$	$+$	$\text{sign}\left(\left(\left(\sigma^2\right)'(0) - L/\gamma \cdot \mu''(0)\right) \cdot s\right)$
$\pi^\rho(s)$	$+^*$	$\text{sign}\left(\left(\left(\sigma^2\right)'(0) - \gamma/L \cdot \kappa_3(0)\right) \cdot s\right)$
$\pi^\iota(s)$	$+^*$	$\text{sign}\left(\left(\left(\sigma^2\right)'(0) - L/\gamma \cdot \mu''(0)\right) \cdot s\right)$
$\tau^\rho(s)$	$+$	$\text{sign}\left(\left(\left(\sigma^2\right)'(0) - \gamma/L \cdot \kappa_3(0)\right) \cdot s\right)$
$\tau^\iota(s)$	$+$	$\text{sign}\left(\left(\left(\sigma^2\right)'(0) - L/\gamma \cdot \mu''(0)\right) \cdot s\right)$

for all y if marked by $*$, and for $s : |s| < x_0$ otherwise.

2) Local price reaction function asymmetry:

$$\text{sign}(\Pi(s) - \Pi(-s)) = \text{sign}\left(2\gamma L \left(\sigma^2\right)'(0) - \gamma^2\kappa_3(0) - L^2\mu''(0)\right) \; \forall s : |s| < x_0.$$

One additional force determines the shape of the liquidity measures relative to the baseline model. In the setting considered here, the moments are *functions* of the supply s. In addition to higher moments, the shape of the conditional moments (as functions of the size of the supply s) also plays a role. In particular, the convexity of the information component of the liquidity measures is also driven by the convexity of the conditional mean function, whereas the convexity of the risk component is driven by the slope of the conditional variance. Note that one would not capture those effects in the jointly normal setting (see, e.g., example 1) because in that case the conditional mean is linear (convexity is zero) and the conditional variance is a constant (the slope is zero).

If one can separate the the inventory risk and the information components of the price impact (Muraviev (2015) provides an excellent example of how one can do so), then one can extract the information about how an order affects the expectations of traders accommodating it. The proposition below shows that the slope of the information component of the price impact is proportional to the slope of the conditional mean $\mu(s)$.

Proposition 8. *The slope of the conditional mean function and the slope of the information component of the price impact are related as follows: $\mu'(0) = -\frac{(\Pi^\iota)'(0)}{L(1+\nu)}$.*

The slope of the conditional mean function shows how a marginal unit liquidated by a block trader affects the expectations of liquidity providers regarding the asset's payoff. Consequently, all else being equal, in markets with greater slope of $\Pi^\iota(\cdot)$, a marginal liquidated unit affects the expectation more, which is indicative of a more informed block trader.

6 Empirical evidence and testable predictions

I begin by summarizing the key findings concerning the shape of the price impact. These are as follows:

1. The price reaction function is a concave function of the order size.[21]

2. The price reaction function is an asymmetric function: the price impacts of sell and buy orders are different.[22]

Proposition 3 implies that with bounded asset payoff, the price reaction function is concave for a large order size, providing the explanation for the first finding. Propositions 3 and 7 link the asymmetry of the permanent price impact to asymmetry and skewness of the distribution. The model predicts that with positive skewness (a natural property for stocks at the individual level (e.g. Chen et al. (2001))) price impact of small purchases is greater than that of small sells, consistent with evidence summarized by Saar (2001).

While Saar (2001) summarizes the evidence of a greater price impact of purchases compared to sells, Chiyachantana et al. (2004) find that the asymmetry of the permanent price impact is linked to the underlying market condition, and show, in particular, that in the bearish markets sells have a higher price impact. My model links the local asymmetry to skewness, and predicts that when skewness is negative small sells have a higher price impact compared to small purchases. Perez-Quiros and Timmermann (2001) present the evidence that skewness varies with the underlying market condition. The model predictions are therefore in line with these findings.

While empirical papers typically find concave price reaction functions, there is some conflicting evidence for price impact of purchases. E.g., Keim and Maddavan (1996) and Holthausen, Leftwich and Mayers (1990) find the convex price reaction function for purchases. This is in line with the prediction of my model: with positive skewness the price reaction function for small purchases is convex. Thus, model findings provide a way of reconciling the conflicting evidence.

Proposition 8 demonstrates that the slope of the asymmetric information component of the price reaction function is linked to the slope of the conditional mean function (i.e., the extent to which, on the margin, the order affects the expectations of liquidity providers about the asset's payoff). Muraviev (2015) presents a methodology that makes it possible to separate the price impact into inventory and information components. The idea is that, when assets are traded on multiple exchanges, the information spreads among traders on different exchanges, while the inventory risk is only accommodated by the traders on the exchange in which the block trader executes the order. Consequently, one can estimate the shape of the information component of the price impact, in particular its slope, which is diagnostic of the informativeness of block trades and can be useful in detecting informed trading.

My model identifies two forces that shape the price reaction function. For small order sizes, its' curvature is linked to skewness. The model implies that the difference in curvatures of price reaction function for purchases and sells is positive with positive skewness and is negative with negative skewness. One can test it in equities market. At the individual level stock returns are positively skewed, whereas at the aggregate level the skewness is negative. Therefore, the difference in curvatures of the price reaction function for purchases and sells should be positive for individual stocks but negative for ETFs.

For large order sizes, the bounds of the payoff represent another force affecting the curvature of price reaction function. One can examine the role of this force in the options market. For put options, the

[21] Equities: Hausman et al. (1992), Almgren et al. (2005), Frazzini et al. (2014). Options: Muraviev (2015).

[22] Saar (2001) summarizes the evidence that shows bigger price impact of buy orders compared to sell orders. However, Chiyachantana et al. (2004) link the asymmetry to the underlying market condition and find that in bullish markets buy orders have a bigger price impact than sells, while in the bearish markets sells have a higher price impact.

upside is limited, while this is not the case for calls. Consequently, for buy orders, the price reaction function should be more concave for puts relative to calls. Muraviev (2015) finds that the price reaction functions in options markets are concave, but, unfortunately, he did not estimate them separately for calls and puts.

One can also examine the role of payoff bounds and skewness by running cross-sectional regressions, in which one estimates the non-linear model of price reaction function also controling for skewness and "distance to the bound", maximal payoff minus current price for purchases and current price minus minimal payoff for sells.

The model also implies that the curvature and asymmetry is more pronounced when market is less competitive and when the risk bearing capacity is smaller. Following Nagel (2012) the changes in risk bearing capacity can be proxied by VIX. Consequently, when VIX is high, the asymmetry and curvature of price response should be higher. The variation in competitiveness can be observed by comparing after hours and regular hours market. There are less market makers in the after hours market and, consequently, the asymmetry and difference in curvatures should be higher. At a higher frequency, the competitiveness of the market can be proxied by Herfindahl indices, as in Hasbrouck (2015).

7 Conclusion

This paper presents a tractable model of strategic trading without normality. It develops a methodology that makes it possible to solve for the equilibrium in a constructive way, which allows one to uncover the multiplicity of equilibria. Closed-form solutions are helpful in selecting the unique equilibrium. The paper also demonstrates that the two forces absent under the normal distribution, the boundedness of the support and higher moments, play an important role in determining the shape of liquidity measures, such as price reaction function and its' components.

This paper focuses on the implications of departures from normality to shape of price impact function and shuts down several channels that might be worth studying. First, as all large traders are symmetric, there is no risk sharing between them. Second, as all traders are informed symmetrically, no information aggregation is taking place. Finally, as the model is static, I am unable to study how nonlinearities in the shape of the liquidity measures affect optimal order break up and the dynamics of the price impact.

Extending the model in any of the above directions is promising and is left for the future work. Below, I comment on the potential difficulties that one may encounter in investigating in those directions.

Solving the model with heterogeneous traders to study risk sharing is challenging because the symmetry of equilibrium generating the tractability in this paper will no longer be present.

Incorporating asymmetric information into the model is challenging because without normality, the bids are not linear and the uncertainty faced by each trader (which comes from the signals that other traders receive and the supply) does not aggregate in a scalar parameter shifting the residual supply curve, and one cannot use ex post maximization techniques.

The dynamic extension is challenging for a reason similar to the heterogeneous case. One has to find a value function, which will depend on endowments of other traders. In particular, one should find the value function for the case in which the endowments are asymmetric.

Despite the above mentioned difficulties, I believe that it still might be possible to approach the above questions, perhaps with the help of numerical techniques.

A Omitted proofs.

Proof. (Theorem 1) Consider a particular trader and fix the strategies of all other traders to be the equilibrium bid $x(p)$. Denote the inverse of $x(p)$ by $I(x)$. It is proved in the Appendix C that the equilibrium bids are strictly decreasing and have a finite slope. Given this the inverse residual supply and the inverse bid are both well-defined objects. The inverse residual supply is given by

$$P(x;s) = I\left(\frac{s-x}{L-1}\right). \tag{31}$$

The ex-post maximization problem $\mathcal{P}_{EP}$ can be written as

$$\max_{x,p} f(x) - p \cdot x$$

$$s.t.:\ p = P(x;s).$$

Substituting the constraint into the objective and taking the first order condition with respect to x yields the following equation determining optimal quantity $x^*(s)$ on the residual supply curve for a given realization of s

$$f'(x^*) - x^* \underbrace{\frac{\partial}{\partial x} P(x;s)\,|_{x=x^*}}_{=\lambda(x^*)} - P(x^*;s) = 0. \tag{32}$$

Since $P(x^*(s);s)$ is the optimal price p^* corresponding to x^* the above becomes the expression (13) for the inverse bid

$$f'(x^*) - \lambda(x^*)\, x^* = p^* = I(x^*).$$

Differentiating the above with respect to x^* and applying the link (11) between the price impact function and the slope of the inverse bid in the symmetric equilibrium I get the ODE (29)

$$x\lambda'(x) = (L-2)\lambda(x) + f''(x). \tag{33}$$

The Proposition C1 implies that $0 < \lambda(x) < \infty$. The only thing that left is to check the second order conditions.

Second order conditions.

I will verify that 1) the first order condition gives unique candidate $x^*(s) = \frac{s}{L}$ and 2) the second order conditions are satisfied for this x^*.

First, plugging (31) into (32) one rewrites the first order condition (32) as

$$f'(x) + \frac{x}{L-1} I'\left(\frac{s-x}{L-1}\right) = I\left(\frac{s-x}{L-1}\right). \tag{34}$$

In the symmetric equilibrium $x = s/L$ should be optimal, so plugging it into (34) I find the ODE for the inverse bid (I denote $y = s/L$)

$$f'(y) + \frac{y}{L-1} I'(y) = I(y). \tag{35}$$

To show 1) I need to demonstrate that given that inverse bid satisfies (35) the unique solution to (34) is $x^* = s/L$.

Define $\xi = \frac{s-x}{L-1}$ and rewrite (34) as

$$f'(x) + \frac{x}{L-1} I'(\xi) = I(\xi).$$

For a given ξ the function on the left hand side of the above decreases in x whereas the right hand side does not depend on x. Therefore for each ξ there is at most one $x^*(\xi)$ solving the above. Since $x^*(\xi) = \xi$ is a solution (the above becomes (35)) we find that x^* solving $x^* = \frac{s-x^*}{L-1}$ is the unique optimal quantity. Therefore $x^* = s/L$.

To check second order conditions at $x^* = s/L$ differentiate (34) with respect to x to get the expression for the second derivative of the objective function

$$\begin{aligned}
\text{second derivative} &= f''(x) - \frac{2}{L-1} I'\left(\frac{s-x}{L-1}\right) - \frac{x}{(L-1)^2} I''\left(\frac{s-x}{L-1}\right) \\
&= f''(x^*) - \frac{2}{L-1} I'(x^*) - \frac{x^*}{(L-1)^2} I''(x^*) \\
&= f''(x^*) - \lambda(x^*) + \frac{x^* \lambda'(x^*) - (L-1)\lambda(x^*)}{(L-1)} \\
&= (f''(x^*) - \lambda(x^*))\left(1 + \frac{1}{L-1}\right) < 0,
\end{aligned}$$

where the second line substitutes $x^* = s/L$, the third line uses (11), the fourth line substitutes (33). Since the second derivative is negative, the second order conditions are satisfied. □

Proof. (Proposition 1) (1) The equilibrium exists iff there is a solution to (14) satisfying $0 < \lambda(x) < \infty$. It follows from (17) that

$$x^{2-L}\lambda(x) = x_0^{2-L}\lambda(x_0) + \int_{x_0}^{x} f''(t) t^{1-L} dt. \tag{36}$$

Consider the case $x \geq x_0 > 0$. The condition $0 < \lambda(x) < \infty$ should hold for all x, therefore $\int_{x_0}^{x} f''(t)t^{1-L}dt$ has to be a bounded function of x for a given x_0. Indeed, we have that $-x_0^{2-L}\lambda(x_0) < \int_{x_0}^{x} f''(t)t^{1-L}dt$ from $\lambda(x) > 0$ and $\int_{x_0}^{x} f''(t)t^{1-L}dt < 0$ from $f''(t) < 0$.

Consider a function $\phi(x) = \int_{x_0}^{x} f''(t)t^{1-L}dt$. It is a decreasing function, which according to the above should be bounded. The latter condition is equivalent to $\lim_{x\to\infty} \phi(x) < \infty$.Thus we have that

$$\lim_{x\to\infty} \int_{x_0}^{x} f''(t)t^{1-L}dt < \infty,$$

which is equivalent to

$$\lim_{x\to\infty} \underbrace{x_0^{L-2} \int_{x_0}^{x} f''(t)t^{1-L}dt}_{(2-L)\rho_{1+\nu}(x_0)/\gamma} < \infty, .$$

The above implies that $\rho_{1+\nu}(x_0) < \infty$ for any given $x_0 > 0$. We can show that $\rho_{1+\nu}(x_0) < \infty$ $\forall x_0 < 0$ analogously. The fact that $\rho_{1+\nu}(0) < \infty$ follows from $\rho_{1+\nu}(0) = \sigma^2$.

The fact that $\rho_{1+\nu}(x_0) < \infty$ for any distribution with bounded support follows from the Fact 5.

(2) I first find a particular solution and then find the general solution by adding the solution of the homogenous equation. Consider the solution satisfying $\lambda(x) = o\left(x^{L-2}\right)$. Taking the limit in the (36) as $x \to \infty$we get that this solution is given by

$$\lambda(x_0) = \nu\gamma\rho_{1+\nu}(x_0). \tag{37}$$

It follows from the above that $\frac{\lambda(x_0)}{x_0^{L-2}} = -\int_{x_0}^{\infty} f''(t)t^{1-L}dt$. The limit of the latter expression as $x_0 \to \infty$ is zero, so $\lambda(x)$ is indeed $o\left(x^{L-2}\right)$.

The general solution to the ODE (14) can be written as a particular solution plus a general solution of a homogenous ODE. Hence one can write

$$\lambda(x) = \nu\gamma\rho_{1+\nu}(x) + 1(x \geq 0)C^{+}x^{L-2} + 1(x < 0)C^{-}\left(-x\right)^{L-2}.$$

Since $\rho_{1+\nu}(x) = o\left(x^{L-2}\right)$ only solutions with $C^{+}, C^{-} \geq 0$ will satisfy $\lambda(x) > 0$. They all will satisfy $\lambda(x) < \infty$ because $\rho_{1+\nu}(x) < \infty$.

(3) The equilibrium inverse bid follows directly from the condition (2) of the Theorem 1. □

Proof. (Proposition 2) When the supply realization is s the equilibrium price in the case $C^{+} = C^{-} = 0$ is

$$p = I(s/L) = g'(-\gamma s/L) - \nu\gamma\rho_{1+\nu}(s/L)s/L.$$

The bid is strictly decreasing and it follows from Results B2 and B4 that its limit as $x \to -\infty(x \to +\infty)$ is equal to $b(a)$. The inverse bid and prices are thus within $[a, b]$. If $C^{+} > 0$ ($C^{-} > 0$) then the inverse bid is less than a (greater than b) for high enough (low enough) s due to unbounded power terms $C^{+}x^{L-2}$ (respectively $C^{-}x^{L-2}$). This fact implies that the equilibrium satisfying property (1) is unique.

To prove (2) note that one can write

$$p_n(s) = \int_1^{\infty} z^{-L} g_n'(-\gamma z \cdot s/L)dz,$$

where $g_n(t)$ is a CGF of an asset that pays δ_n. Interchanging the limit and integration (which is possible due to Dominated Convergence Theorem) one gets

$$p_n(s) \to \int_1^{\infty} z^{-L} \lim_{\substack{a_n \to -\infty \\ b_n \to +\infty}} g_n'(-\gamma z \cdot s/L) \cdot dz = \int_1^{\infty} z^{-L} g'(-\gamma z \cdot s/L) \cdot dz,$$

where $g(t)$ is a CGF of an asset that pays δ. The last equality follows from the application of the Dominated Convergence Theorem.

The third property follows directly by taking the limits in the results of Proposition 1. □

Proof. (Proposition 3) The monotonicity of price reaction function follows from the monotonicity of equilibrium bid (recall that $p(s) = I(s/L)$). The monotonicity of permanent price impact in $|\bar{s}|$ follows from

$$\pi(s) = |f'(\bar{s}) - \mu|, \tag{38}$$

eq. (2) and the convexity of the CGF (Fact 1 in the Appendix B).

That $\pi(-\infty) = b - \mu$ and $\pi(+\infty) = \mu - a$ follows from (38), (2) and Fact 2.

The limits $\tau(-\infty) = 0$ and $\tau(+\infty) = 0$ follow from (22) and the Fact 5.

Since $\Pi(\cdot) = \pi(\cdot) + \tau(\cdot)$, one gets $\Pi(-\infty) = b - \mu$ and $\Pi(+\infty) = \mu - a$ immediately from the above two limits.

It can be shown that the inverse bid is given by $I(x) = \int_1^\infty z^{-L} f'(zx)dz$. The convexity of the price reaction function at x is the same as the convexity of the inverse bid x/L. The latter is given by

$$I''(x) = \int_1^\infty z^{2-L} f'''(zx)dz.$$

For large enough x, the term $f'''(zx)$, $z \geq 1$ is either positive or negative, and hence the bid is either concave or convex. The convex shape is ruled out by the fact that the inverse bid is bounded. □

Proof. (Proposition 4) The global monotonicity of the functions $\Pi(\cdot)$ and $\tau(\cdot)$ was established in the Proposition 3 above. To prove that $\tau(|s|)$ is locally increasing I calculate its' derivative at zero. For positive s the function $\tau(|s|)$ can be written as (cf. (22)):

$$\tau(|s|) = \tau(s) = \nu\gamma\rho_{1+\nu}(\bar{s})\bar{s}.$$

Calculating the derivative with respect to $\bar{s}$ at zero one gets (cf. Fact 4) $\nu\gamma\rho_{1+\nu}(0) = \nu\gamma\sigma^2 > 0$. For negative s we have $\tau(|s|) = \tau(-s) = -\nu\gamma\rho_{1+\nu}(-\bar{s})\bar{s}$ and the calculation leads, analogously $\tau' = \nu\gamma\sigma^2 > 0$. Calculating the second derivative of $\tau(\cdot)$ at zero one gets $\nu\gamma\rho'_{1+\nu}(0) = -\nu\gamma\kappa_3\sigma^3$, where I've used Fact 4 and (1). For negative s the calculation is analogous and yields $\nu\gamma\kappa_3\sigma^3$. We thus get $\tau''(0+) = -\text{sign}(\kappa_3)$ while $\tau''(0-) = \text{sign}(\kappa_3)$. The convexity of price reaction function and the permanent price impact is considered analogously.

To prove the result about asymmetry note that the permanent price impact function for buy and sell orders have common slope at the origin, but different convexity. The asymmetry then follows.

The comparative statics follow from the following approximation, which one can obtain using Fact 3 and Fact 4: $x\rho_a(x) = \sigma^2 x - \frac{2\alpha}{1+\alpha}\gamma g'''(0)x^2 + o\left(x^2\right)$. □

Proof. (Theorem 2) Consider trader i and let his bid be $x_i(p)$. Fix the strategies of all other traders to be equilibrium bid $x(p)$. The residual supply in that case is given by

$$R(p; s) = s - (L-1)x(p). \tag{39}$$

The ex-post maximization problem $\mathcal{P}_{EP}$ can be written as

$$\max_{x,p} f(x,s) - p \cdot x$$
$$s.t.: \ x = R(p; s).$$

Substituting the constraint into the objective and taking the first order condition with respect to p yields the following equation determining optimal price $p^*(s)$ on the residual supply curve for a given realization of s

$$f_1(R(p^*, s), s)R_p(p^*, s) - R(p^*, s) - p^* \cdot R_p(p^*; s) = 0. \tag{40}$$

Given that the second-order conditions are satisfied the optimal price $p^*(s)$ is unique. The corresponding optimal quantity $x^*(s)$ is given by $x^*(s) = R(p^*; s)$. In the symmetric equilibrium $x^*(s) = x(p^*(s)) = s/L$. It follows that $R(p^*; s) = b(p^*(s))$. Substituting it to (40), noting that $R_p(p^*; s) = -(L-1)b'(p^*(s))$ (follows directly from (39)) and denoting $p = p^*(s)$ we rewrite (40), after some rearrangement, as

$$b'(p) = \frac{b(p)}{(L-1)(p - f_1(b(p), Lb(p)))} \equiv \phi(p, b). \tag{41}$$

Under assumption 1, $\psi(x) = f_1(x, Lx)$ is strictly decreasing in x. The analysis of the symmetric info case is applicable substituting $\psi(x)$ instead of $f'(x)$. In particular the ODE for price impact function is obtained by substituting $\psi'(x)$ instead of $f''(x)$ in the ODE (14) the analysis of the second order conditions can be done via the analogous substitution. □

Proof. (Proposition 5) The proof follows the same steps as the proof of Proposition 1 writing $h(x)$ instead of $f''(x)$ everywhere. □

Proof. (*Proposition 6)* The monotonicity of price reaction function follows from the monotonicity of equilibrium bid. The monotonicity of the permanent price impact follows from $\pi(s) = |f_1(\bar{s}, L\bar{s}) - \mu|$ and the fact that the function $f_1(\bar{s}, L\bar{s})$ is monotone in $\bar{s}$ provided that Assumption 1s holds.

To get the limits of $\pi(\cdot)$ at infinity note two facts. First, for any fixed y the we have$a \leq f_1(x, y) \leq b$ (monotonicity of $f_1(x, y)$ in x and the Fact 2). Second, Assumption 1s implies that for any $y \geq 0$ $f_1(x, y) \leq f_1(x, 0)$. Combining the two facts we get that $a \leq f_1(x, Lx) \leq f_1(x, 0)$. Taking the limit as $x \to +\infty$ using Fact 2 and applying Squeezing Theorem we get that $\lim_{x\to+\infty} f_1(x, Lx) = a$ from which $\pi(+\infty) = \mu - a$ follows. The limit at minus infinity can be derived analogously.

To get the limits of price reaction function at infinity note that it can be written as $\Pi(s) = |p(s) - \mu| = \left|(L-2)\int_1^\infty f_1(\bar{s}t, L\bar{s}t)t^{1-L}dt - \mu\right|$ (This can be most easily derived by solving the ODE $xI'(x) = (L-2)I(x) - f_1(x, Lx)$ for the inverse bid. The ODE can be obtained from 29 using $\lambda(x) = -\frac{1}{L-2}I'(x)$.). Now calculate the limit of $(L-2)\int_1^\infty f_1(xt, Lxt)t^{1-L}dt$ as $x \to \infty$. Using the Monotone Convergence Theorem and $\lim_{x\to+\infty} f_1(x, Lx) = a$ derived above we can write $\lim_{x\to+\infty}(L-2)\int_1^\infty f_1(xt, Lxt)t^{1-L}dt = (L-2)\int_1^\infty \lim_{x\to+\infty} f_1(xt, Lxt)t^{1-L}dt = a(L-2)\int_1^\infty t^{1-L}dt = a$. The limit at minus infinity can be derived analogously.

The limit of $\tau(\cdot)$ at infinity follows from $\tau(\cdot) = \Pi(\cdot) - \pi(\cdot)$ and the limits derived above. □

Proof. (Proposition 7 and 8) I start by deriving the results for the permanent price impact. The risk part for $s > 0$ can be written as

$$\pi^\rho(s) = \gamma\bar{s}\int_0^1 g_{11}(-\gamma t\bar{s}, Lt\bar{s})dt = \gamma\int_0^{\bar{s}} g_{11}(-\gamma y, Ly)dy.$$

Its derivative is thus given by $\gamma g_{11}(-\gamma\bar{s}, L\bar{s}) > 0$ and the second derivative at zero with respect to $|\bar{s}|$ is given by $-\gamma^2 g_{111}(0,0) + \gamma L g_{112}(0,0) = -\gamma^2 k_3(0) + \gamma L\left(\sigma^2\right)'(0)$ for $s > 0$ with the sign flipped for $s < 0$.

The information part can analogously be written as

$$\pi^\iota(s) = -L\int_0^{\bar{s}} g_{12}(-\gamma y, Ly)dy.$$

Its' first derivative is given by $-L \cdot g_{12}(-\gamma\bar{s}, L\bar{s}) > 0$ (Assumption 1s) (equal to $-L\mu'(0)$ at zero) and the second derivative at zero is $\gamma L\sigma'(0) - L^2\mu''(0)$ for $s > 0$ with the sign flipped for $s < 0$.

The risk part of the strategic component of the price reaction function can be written as

$$\tau^{\rho}(s) = \nu\gamma\bar{s}\int_0^1 g_{11}(-t^{-\nu}\gamma\bar{s}, Lt^{-\nu}\bar{s})dt.$$

It's first derivative at zero is given by $g_{11}(0,0) = \nu\gamma\sigma^2(0) > 0$ the second derivative at zero is given by $2\nu\gamma\int_0^1 t^{-\nu}\left(Lg_{112}(0,0) - \gamma g_{111}(0,0)\right)dt = \frac{2\nu}{1-\nu}\gamma L(\left(\sigma^2\right)'(0) - \gamma/L \cdot \kappa_3(0))$ for $s > 0$ with the sign flipped for $s < 0$.

The information part of the strategic component of the price reaction function can be written as

$$\tau^{\iota}(s) = -\nu L\bar{s}\int_0^1 g_{12}(-t^{-\nu}\gamma\bar{s}, Lt^{-\nu}\bar{s})dt.$$

Its' first derivative at zero is given by $-\nu Lg_{12}(0,0) > 0$ (equal to $-\nu L\mu'(0)$ at zero). The second derivative at zero is given by $-2\nu L\int_0^1 t^{-\nu}\left(Lg_{122}(0,0) - \gamma g_{112}(0,0)\right)dt = -\frac{2\nu}{1-\nu}\gamma L(L/\gamma \cdot \mu''(0) - \left(\sigma^2\right)'(0))$ for $s > 0$ with the sign flipped for $s < 0$.

The results for the price reaction function are obtained combining the results for $\pi(\cdot)$ and $\tau(\cdot)$. In particular, the slope of the price reaction function is given by $-L(1+\nu)\mu'(0)$. $\square$

B Some properties of CGFs and the risk function

I start by outlining some properties of CGFs. They are certainly known but I present them with proofs to provide a self-contained treatment of the results.

B.1 Properties of CGFs

Fact 1. *The CGF of a non-degenerate distribution is strictly convex.*

Proof. Differentiating the definition of the CGF twice one gets

$$g''(x) = \frac{E[\delta^2\exp(\delta x)]E[\exp(\delta x)] - E[\delta\exp(\delta x)]^2}{E[\exp(\delta x)]^2}.$$

The sign of $g''(x)$ is equal to the sign of $E[\delta^2\exp(\delta x)]E[\exp(\delta x)] - E[\delta\exp(\delta x)]^2$. To complete the proof apply the Cauchy-Schwartz inequality (stating that $E[XY]^2 < E\left[X^2\right]E\left[Y^2\right]$ for linearly independent random variables X and Y) to the random variables $X = \delta\exp(\delta x/2)$ and $Y = \exp(\delta x/2)$. $\square$

The next result is particularly important to analyze the case of δ having bounded support.

Fact 2. *Suppose that the support of δ is (a,b) (with a and b possibly infinite). The first derivative of the CGF, $g'(x)$ is increasing and*

$$\lim_{x\to-\infty} g'(x) = a \text{ and } \lim_{x\to+\infty} g'(x) = b.$$

This result relies on the following two Lemmas.

Lemma 1. *Suppose that the support of δ is (a, b) (with a and b possibly infinite). Consider $c : a < c < b$. Then $E[\delta e^{x\delta}\mathbb{I}(\delta < c)] = O\left(e^{cx}\right)$ as $x \to +\infty$.*

Proof. We need to show that $\exists x_0$, M such that $\forall x > x_0$

$$|E[\delta e^{x\delta}\mathbb{I}(\delta < c)]| < Me^{xc}. \tag{42}$$

Without loss of generality consider $x > 0$. Rewrite $|E[\delta e^{x\delta}\mathbb{I}(\delta < c)]|$ as

$$\begin{aligned}
|E[\delta e^{x\delta}\mathbb{I}(\delta &< c)]| = Pr(\delta < c) \cdot |E[\delta e^{x\delta}|\delta < c]| \\
&< |c|E[e^{x\delta}|\delta < c] \\
&< |c|e^{xc}.
\end{aligned}$$

It is clear that (42) holds with $M = |c|$ and $x_0 = 0$. □

Lemma 2. *Suppose that the support of δ is (a, b) (with a and b possibly infinite). Consider $c : a < c < b$. There exists $K > 0$ and $\theta > c$ such that*

$$E[e^{x\delta}\mathbb{I}(\delta \geq c)] \geq Ke^{\theta x}. \tag{43}$$

Proof. Rewrite $E[e^{x\delta}\mathbb{I}(\delta \geq c)]$ as

$$E[e^{x\delta}\mathbb{I}(\delta \geq c)] = Pr(\delta \geq c) \cdot E[e^{x\delta}|\delta \geq c].$$

By Jensen's inequality $E[e^{x\delta}|\delta \geq c] \geq e^{x\theta}$, where $\theta = E[\delta|\delta \geq c] > c$. Clearly, (43) holds with $K = Pr(\delta < c)$ and $\theta = E[\delta|\delta \geq c]$. □

I am now ready to provide the prof for the Fact 2.

Proof. (Fact 2) That $g'(x)$ is increasing follows directly from the Fact *1*.

I only prove that $\lim_{x\to+\infty} g'(x) = b$, since the second limit is analogous.

Consider arbitrary $c : a < c < b$ and rewrite

$$\begin{aligned}
g'(x) &= \frac{E[\delta e^{x\delta}]}{E[e^{x\delta}]} = \frac{E[\delta e^{x\delta}\mathbb{I}(\delta < c)]}{E[e^{x\delta}\mathbb{I}(\delta < c) + E[e^{x\delta}\mathbb{I}(\delta \geq c)]} \\
&+ \frac{E[\delta e^{x\delta}\mathbb{I}(\delta \geq c)]}{E[e^{x\delta}\mathbb{I}(\delta < c) + E[e^{x\delta}\mathbb{I}(\delta \geq c)]},
\end{aligned}$$

Consider the first term. It follows from Lemmas 1 and 2 that there exist K, $M > 0$ and $\theta > c$ such that

$$\left|\frac{E[\delta e^{x\delta}\mathbb{I}(\delta < c)]}{E[e^{x\delta}\mathbb{I}(\delta < c) + E[e^{x\delta}\mathbb{I}(\delta \geq c)]}\right| \leq \frac{Me^{xc}}{Ke^{x\theta}} \to 0 \text{ as } x \to \infty.$$

It then follows from a Squeezing Theorem that the limit of the first term is zero. The latter implies that

$$\lim_{x\to\infty} g'(x) = \lim_{x\to\infty} \frac{E[\delta e^{x\delta}\mathbb{I}(\delta \geq c)]}{E[e^{x\delta}\mathbb{I}(\delta < c) + E[e^{x\delta}\mathbb{I}(\delta \geq c)]}.$$

The above and the fact that $cE[e^{x\delta}\mathbb{I}(\delta \geq c] < E[\delta e^{x\delta}\mathbb{I}(\delta \geq c] < bE[e^{x\delta}\mathbb{I}(\delta \geq c]$ imply that

$$c \leq \lim_{x\to\infty} g'(x) \leq b.$$

Because c is arbitrary and the limit cannot depend on c, we find that the limit is b. □

B.2 Properties of the risk function

Now I present the properties of the risk function $\rho_a(x)$. In all derivations I will assume that $x \geq 0$. The relevant derivations for the case $x \leq 0$ can be obtained analogously.

I will denote

$$\alpha = \frac{1}{1-a}.$$

I will also maintain the assumption

$$a \geq 0,\ a \neq 1$$

throughout this section which clearly holds for the cases $a = 0$ and $a = 1+\nu$ which are of interest for the results of the paper. The case $0 \leq a < 1$ corresponds to $\alpha \geq 1$, whereas the case $a > 1$ corresponds to $\alpha < 0$.

Fact 3. *The risk function $\rho_a(x)$ satisfies the ODE*

$$x\rho_a'(x) = -\alpha\rho_a(x) + \alpha g''(-\gamma x). \tag{44}$$

Proof. Write the definition of the risk function

$$\rho_a(x) = \int_0^1 g''(-t^{1-a}\gamma x)dt.$$

Make a change of variable $t = (y/x)^\alpha$, $dt = \alpha\,(y/x)^{\alpha-1}\,\frac{dy}{x}$. The above becomes

$$\rho_a(x) = \begin{cases} \alpha x^{-\alpha}\int_0^x g''(-\gamma y)y^{\alpha-1}dy, & \alpha \geq 1 \\ -\alpha x^{-\alpha}\int_x^\infty g''(-\gamma y)y^{\alpha-1}dy, & \alpha < 0 \end{cases} \tag{45}$$

Differentiating the above with respect to x one gets (44). □

Fact 4. Provided that $\alpha \neq -n$, the n-th derivative of $\rho_a(x)$ is continuous at zero and is given by

$$\rho_a^{(n)}(0) = \frac{(-\gamma)^n\,\alpha}{n+\alpha} g^{(n+2)}(0).$$

Proof. Rewrite (44) as

$$\rho_a'(x) = -\alpha\left(\frac{\rho_a(x)-\sigma^2}{x} + \gamma\frac{g''(-\gamma x)-\sigma^2}{-\gamma x}\right).$$

Calculating the limit of the above as $x \to 0$ we get

$$\rho_a'(0) = -\alpha\rho_a'(0) - \alpha\gamma g'''(0).$$

The above implies

$$\rho_a'(0) = \frac{-\gamma\alpha}{1+\alpha} g'''(0).$$

So if $\alpha \neq -1$ $\rho_a'(x)$ is continuous at zero.

To get the second derivative at zero differentiate (44)

$$x\rho_a''(x) = -(1+\alpha)\rho_a'(x) - \gamma\alpha g'''(-\gamma x).$$

Rewrite the above as

$$\rho_a''(x) = -(1+\alpha)\left(\frac{\rho_a'(x)-\rho_a'(0)}{x} - \frac{\gamma^2\alpha}{1+\alpha}\frac{g'''(-\gamma x)-g'''(0)}{-\gamma x}\right).$$

Taking the limit as $x \to 0$ we get

$$\rho_a''(0) = -(1+\alpha)\rho_a''(0) + \gamma^2\alpha g^{(4)}(0).$$

We get

$$\rho_a''(0) = \frac{\gamma^2\alpha}{2+\alpha} g^{(4)}(0).$$

Again, if $\alpha \neq -2$ $\rho_a''(x)$ is continuous at zero. One can get the general formula by induction:

$$\rho_a^{(n)}(0) = \frac{(-\gamma)^n\alpha}{n+\alpha} g^{(n+2)}(0).$$

□

Fact 5. *Suppose δ has bounded support $[a, b]$. Then for $a > 1$: $\rho_a(x) < \infty$ and*

$$\rho_a(x) = o(1/x) \text{ as } x \to \pm\infty.$$

Proof. First I prove that $\rho_a(x) < \infty$. According to (45) for $a > 1$ (i.e. $\alpha < 0$) we can write the risk function as

$$\rho_a(x) = -\alpha x^{-\alpha}\int_x^\infty g''(-\gamma y) y^{\alpha-1} dy.$$

Integrating by parts one gets

$$\rho_a(x) = \frac{-\alpha}{\gamma x}\left(g'(-\gamma x) - (\alpha-1)\, x^{1-\alpha}\int_x^\infty g'(-\gamma y) y^{\alpha-2} dy\right). \tag{46}$$

Rearranging the above I get

$$\rho_a(x) = \frac{\alpha(\alpha-1)}{\gamma}\left(\int_1^\infty \left(\frac{g'(-\gamma x) - g'(-\gamma x t)}{x}\right) t^{\alpha-2} dt\right).$$

Since $0 < g'(-\gamma x) - g'(-\gamma x t) < b - a$ (Fact 1 and Fact 2) we have that $\rho_a(x) < -\frac{\alpha}{\gamma}\frac{b-a}{x} < \infty$ for $x \neq 0$. To calculate $\rho_a(x)$ at zero we use the Dominated Convergence Theorem to interchange the limit and integration and L'Hopital's rule to calculate $\lim_{x\to 0} \frac{g'(-\gamma x) - g'(-\gamma x t)}{x} = -\gamma\sigma^2(1-t)$. Calculating $\alpha(\alpha-1)\left(\int_1^\infty \sigma^2 (t-1) t^{\alpha-2} dt\right)$ yields σ^2 which is finite (since by assumption the CGF exists for all $x \in \mathbb{R}$.

Now I prove that $\rho_a(x) = o(1/x)$ *as* $x \to +\infty$. We can write $\lim_{x\to\infty} x\rho_a(x) = \frac{\alpha(\alpha-1)}{\gamma}\left(\int_1^\infty \lim_{x\to\infty} (g'(-\gamma x) - g'(-\gamma x t)) t^{\alpha}\right.$
0, where the Dominated Convergence Theorem was used to interchange limit and integration and Fact 2 to calculate $\lim_{x\to\infty} g'(-\gamma x) = \lim_{x\to\infty} g'(-\gamma x t) = a$. □

Fact 6. *The risk function satisfies the following comparative statics results:*

1) $\rho_a(x)$ is homogenous of degree 0 in $(\gamma, 1/x)$;
2) For $a' > a > 1$ and $a' < a < 1$ $1/\alpha' \cdot \rho_{a'}(x) < 1/\alpha \cdot \rho_a(x)$, where $\alpha' = \frac{1}{1-a'}$ and $\alpha = \frac{1}{1-a}$;
3) For two distributions A and B satisfying $g''_A(t) > g''_B(t)$ $\rho_a^A(x) > \rho_a^B(x)$.

Proof. Write the risk function as

$$\rho_a(x) = \begin{cases} \alpha \int_0^1 g''(-\gamma y x) y^{\alpha-1} dy & \text{, if } \alpha > 0; \\ -\alpha \int_1^\infty g''(-\gamma y x) y^{\alpha-1} dy & \text{, if } \alpha < 0. \end{cases} \tag{47}$$

The results are easy to verify from the above. □

C Equilibrium bids are strictly decreasing and have a finite slope

Since we do not know whether the residual supply is invertible, we write the residual supply curve as

$$x = R(p; s).$$

The residual supply is given by

$$R(p; s) = s - (L-1)b(p), \tag{48}$$

where $b(p)$ denotes the equilibrium bid.

The ex-post maximization problem $\mathcal{P}_{EP}$ can be written as

$$\max_{x,p} f(x) - p \cdot x$$

$$s.t.: \; x = R(p; s).$$

Substituting the constraint into the objective and taking the first order condition with respect to p yields the following equation determining optimal price $p^*(s)$ on the residual supply curve for a given realization of s

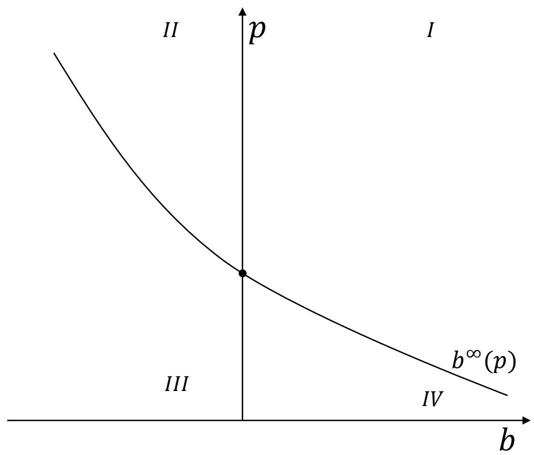

Figure 5: Possible trajectories originating in the II and IV quadrants.

$$f'\left(R\left(p^*;s\right)\right)R_p\left(p^*;s\right)-R\left(p^*;s\right)-p^*\cdot R_p\left(p^*;s\right)=0. \tag{49}$$

The above equation determines the optimal price $p^*(s)$ for a given realization of s. The corresponding optimal quantity x^* is given by $x^*=R(p^*;s)$. In the symmetric equilibrium $x^*=b(p^*)$ (all traders get the same quantity). It follows that $R\left(p^*;s\right)=b\left(p^*\right)$. Substituting it to (49), noting that $R_p\left(p^*;s\right)=-(L-1)b'\left(p^*\right)$ (follows directly from (48)) and denoting $p=p^*(s)$ we rewrite (49), after some rearrangement, as

$$b'(p)=\frac{b}{(L-1)\left(p-f'\left(b\right)\right)}. \tag{50}$$

The nonlinear ordinary differential equation above is a necessary condition for equilibrium bid. The analysis that follows is analogous to Klemperer and Meyer (1989).

The lines $b=0$ and $b=b^\infty(p)$ solving $p-f'\left(b^\infty\right)=0$ divide the (b,p) plane into four quadrants numbered as in the figure below.

Proposition C1. *The equilibrium bid $b(p)$ satisfies$-\infty<b'(p)<0$.*

Proof. The condition $-\infty<b'(p)<0$ is equivalent to saying that the bid lays within the second and fourth quadrants.

Suppose, on the contrary, that the equilibrium bid passes through the first quadrant. Then the price the trader will pay for some realizations of the supply will be above marginal utility $f'(x)$ which cannot be optimal because in that case the trader is strictly better off by submitting $f'(\cdot)$ for the corresponding supply realisations. The bid that passes through the first quadrant cannot be optimal for the same reason. The bid that satisfies $b'(p)=\infty$ is the bid that intersects with the $b=b^\infty(p)$ locus. Such a bid eventually reaches either first or third quadrant and so cannot be optimal. Finally, as follows from (50) the only bid for which $b'(p)=0$ is $b=0$ which is not optimal since the consumer surplus is zero and the deviation to any bid that passes through second and fourth quadrants is profitable. □

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