Persistent Trade and Aggregate Disagreement in Competitive Dynamic Markets*

Efstathios Avdis† Christoph Frei‡ Sergei Glebkin§ Raphael Huywler¶

September 19, 2026

Abstract

Dynamic models with heterogeneously informed investors are hard to solve: investors must forecast the forecasts of others. We overcome this difficulty by constructing a continuum economy as the limit of finite economies in which investors have heterogeneous private signals and agree to disagree about prices. Total signal precision remains finite as the economy grows, and private signals aggregate into a single noisy public signal, revealed by the price, that is a sufficient statistic for others' forecasts. Individual belief distortions vanish in the limit, but their cross-sectional distribution survives as aggregate disagreement and sustains trade. A trade equilibrium exists whenever this disagreement is balanced across investors, and we characterize it without assuming a specific utility function. Market quality and trading activity are jointly shaped by the dispersion of beliefs and wealth inequality, and may increase or decrease in either.

Keywords: Rational Expectations, Trading Activity, Disagreement, Information Aggregation, Risk Aversion, Wealth Heterogeneity, Continuous-Time Equilibrium

1. Introduction

Investors differ in what they know. Asset managers spend heavily on proprietary data—satellite imagery, credit-card records, web traffic—precisely to gain an edge over their competitors. And investors who observe the same data need not draw the same conclusions from

*Efstathios Avdis and Christoph Frei gratefully acknowledge financial support by the Social Sciences and Humanities Research Council of Canada through Insight Grant 435-2022-0507.

†Department of Finance, University of Alberta, Edmonton AB T6G 2R6, Canada; avdis@ualberta.ca.

‡Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton AB T6G 2N8, Canada; cfrei@ualberta.ca.

§Department of Finance, INSEAD, Boulevard de Constance, 77305 Fontainebleau Cedex, France; glebkin@insead.edu.

¶Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton AB T6G 2N8, Canada; huwyler@ualberta.ca.

it: turning raw information into a trading signal requires models and judgment that differ across firms.

Investors also differ in size. Two funds holding identical views will not trade identically. How large a position an investor is willing to take, and how much risk they are willing to bear, depends on how wealthy they are.

How do prices aggregate information across investors who differ in both dimensions? Whose signals end up reflected in the price, and with what weight? And what does the answer imply for trading activity and market quality?

These questions are difficult to answer, for two reasons that mirror the two dimensions of heterogeneity. First, dynamic models with heterogeneously informed investors are intractable outside special cases: as Townsend (1983) puts it, investors must forecast the forecasts of others, and Makarov and Rytchkov (2012) show that the state space then grows without bound. Tractability has been recovered only by restricting the information structure: making private information public after a few periods (e.g., Brown and Jennings (1989), Grundy and McNichols (1989)), ranking investors so that the better informed know everything the less informed know (e.g., J. Wang (1993)), holding the payoff-relevant fundamental constant over time (e.g., He and J. Wang (1995), Foster and Viswanathan (1996)), or letting prices reveal everything (e.g., Kyle, Obizhaeva, and Y. Wang (2017)). Each restriction weakens precisely the channel of interest. Second, the models that do remain tractable rely on constant absolute risk aversion (e.g., J. Wang (1994), Kyle, Obizhaeva, and Y. Wang (2017)), under which wealth has no effect on portfolio choice. Heterogeneity in size then does nothing by construction.

The model. To explore our question, we introduce an economy with perfect competition (infinitely many investors) and speculative trading in a dynamic rational expectations equilibrium without assuming specific utility functions. More specifically, we describe a financial market with a continuum of investors, and two assets: one risk-free and one with Gaussian risk. Each investor has private information, and uses it to trade dynamically in both assets to maximize the expected discounted utility of future consumption.

We construct this economy as a perfectly competitive continuum economy, obtained as the limit of a sequence of finite-agent economies inspired by the information structure of the static CHILE framework of Avdis and Glebkin (2025). In this information structure the precision of each investor's signal vanishes as the market grows, while the total precision of private information stays finite. Each discrete economy has a discretized price, a discretized information structure, and finitely many agents who agree to disagree about the distribution of the price. We implicitly derive the optimal trading strategies and consumption policies in each discrete economy. Then, we aggregate the individual demands and take the limit as the number of investors goes to infinity, obtaining a continuous economy without individual disagreements. We then clear the market and confirm our price conjecture.

This approach has two important properties. First, it generates a type of relaxed rationality with persistent trade that we call "near-rationality".[1] In each discrete economy, investors hold slightly different beliefs about the distributional properties of the price of the risky asset. These belief distortions are small, and they all vanish as the number of investors grows. What survives in the limit is not the distortion of any particular investor, but the way these distortions are distributed across the investor population. When individual demands are aggregated and market clearing is imposed, these small heterogeneous distortions accumulate into a non-zero aggregate term. As a result, the limiting economy has no investor-level disagreement, yet it still inherits enough aggregate disagreement from the approximating discrete economies to sustain equilibrium trade.

Second, our approach allows us to incorporate general preferences because we can derive the Hamilton-Jacobi-Bellman (HJB) equations related to investors' individual utility maximization problems without explicitly solving them. If we had to solve the HJB equations explicitly, we would have to restrict ourselves to constant absolute risk aversion (CARA) utilities because we are working in a Gaussian linear setup. However, because we impose market clearing only in the limiting continuous economy, we do not need explicit closed-form solutions in the discrete (finite) economies. Instead, we exploit the leading-order structure of the HJB equations and pass to the limit, where the resulting equilibrium conditions become tractable.

Contributions. Our contribution is threefold. First, we develop a dynamic model with heterogeneously informed investors that remains tractable, and in which trade arises without exogenous noise supply. Second, this tractability allows us to dispense with constant absolute risk aversion. We characterize equilibrium for general preferences, which is possible because we derive the investors' Hamilton–Jacobi–Bellman equations without having to solve them in closed form. Third, we use this generality to characterize the role of wealth: how it shapes equilibrium holdings, trading activity, and the weight an investor's information receives in the price—a channel that is shut down by construction under CARA.

Information structure. Our setting features continuous and heterogeneous information, following Avdis and Glebkin (2025), whereby the total amount of information in the market remains finite as the economy becomes large (see also Kyle (1989, Section 9)). In a static setting, such an information structure can be modeled by letting the noise in investors' signals be increments of a Brownian motion indexed by the agent. Since ours is dynamic, the noise has to be indexed by time as well as by agent, and we therefore use a Brownian sheet, the two-parameter analogue of Brownian motion (see Adler and Taylor (2007)), which behaves like a scaled Brownian motion in each parameter.

[1] As Milgrom and Stokey (1982) and Tirole (1982) show, strict rationality is not consistent with trade.

Results. Our first result concerns the informational structure of equilibrium. Private signals aggregate into a single noisy public signal, revealed by the price, that is a sufficient statistic for the forecasts of others. The hierarchy of higher-order expectations therefore does not arise, and equilibrium is Markov in the dividend and its public filter.

A second, perhaps less surprising, result is that our model always has a no-trade equilibrium. This result makes sense because the price of the risky asset corresponds to the expected net present value of future dividend payments conditional on public information, and therefore not trading is always a valid strategy in equilibrium.

At the same time, an equilibrium with trade exists if and only if disagreement is sufficiently dispersed.[2] In other words, if all investors had individual beliefs about the distributional properties of the price that go in the same direction, there would not be enough disagreement to generate trade.[3]

A central result of the paper is that prices do not reveal a simple average of private signals. Instead, prices reveal a weighted average of investors' signals with weights determined endogenously by market clearing. In equilibrium, these weights depend explicitly on investors' information precision, risk aversion, and disagreement.

Since we allow for general preferences, the model also characterizes how wealth affects equilibrium trading through its effect on risk aversion. In equilibrium, investors' average risky holdings are proportional to their absolute risk tolerance. Hence, when preferences exhibit decreasing absolute risk aversion, wealthier investors hold larger risky positions and respond more strongly to private information and disagreement. By contrast, under constant absolute risk aversion, this wealth channel is absent. The model therefore nests the CARA benchmark while showing how wealth-dependent risk aversion shapes both risk sharing and trading activity.

We use our model to analyze important "market quality" measures, such as price informativeness and trading activity in a dynamic setting. We show that price informativeness is an affine function of the mean-squared error in the filtering of fundamental information from public information. This relationship means that a lower filtering error is equivalent to prices containing more information.

We measure trading activity by the magnitude of stochastic fluctuations in aggregate demand.[4] We show that aggregate trading activity is proportional to the distance of the equilibrium with trade from the no-trade equilibrium, where this distance is determined endogenously through market clearing and reflects the strength of cross-sectional disagreement. The further the economy deviates from the no-trade benchmark through belief distortions, the greater the reallocation of risky positions across investors, and the higher the trading

[2] Note that as we take the limit the individual level of disagreement vanishes, but its aggregate persists in integrated form.

[3] In the presence of a trade equilibrium, we can measure its distance from a no-trade equilibrium by an endogenous constant obtained from market clearing.

[4] In strict terms, trading activity is defined as the square root of quadratic variation across agents.

activity.

Price informativeness is maximal in the presence of a no-trade equilibrium. More generally, the model yields a rich equilibrium structure in which there is no universal tradeoff between price informativeness and trading activity. Price informativeness is governed by the aggregate precision of the public signal, whereas trading activity is governed by the equilibrium's distance from the no-trade benchmark. These two objects need not move together: for example, scaling beliefs or absolute risk aversion can change trading activity while leaving the equilibrium weight function, and hence price informativeness, unchanged.

We also show that trading activity scales inversely with absolute risk aversion: an increase in risk aversion leads to a proportional decline in aggregate trading activity. This reinforces the wealth channel described above. When absolute risk aversion decreases with wealth, wealthier agents trade more and contribute more strongly to equilibrium volume.

Finally, wealth, price informativeness, and the weight an investor's information receives in the price are not linked monotonically. Wealthier investors' signals need not be reflected more strongly in the price, and redistributing wealth can either raise or lower price informativeness. Which of the two occurs depends on how the induced change in absolute risk aversion interacts with the belief structure, and hence with the equilibrium weight function.

Related literature. Prices that aggregate private information efficiently and contain no noise leave investors with no reason to trade on their information (see Milgrom and Stokey (1982) and Tirole (1982)). A tractable equilibrium with trade therefore usually comes at a cost: either a departure from pure rationality, or the addition of noise. Moreover, closed-form solutions to investors' utility maximization problems in equilibrium are typically available only for particular utility functions.

A common approach to generate trade is to use stochastic supply, where noise traders, whose demand or supply is independent of the market price, match the optimal asset allocation of rational investors (see, e.g., Breon-Drish (2015), Campbell and Kyle (1993), Grossman and Stiglitz (1976), Grossman (1977), Hellwig (1980), J. Wang (1993)). We instead follow the agree-to-disagree approach, in which trade arises because investors hold different beliefs about certain properties of the underlying financial market model (see, e.g., Banerjee, Kaniel, and Kremer (2009), Banerjee and Kremer (2010), Banerjee (2011), Han and Kyle (2018), Kyle, Obizhaeva, and Y. Wang (2017)).

The setup of our financial market is closely related to Kyle, Obizhaeva, and Y. Wang (2017) and J. Wang (1993). Both use linear Gaussian processes to describe dividends and their growth rates, and in both the price is linear in the state variables. Although J. Wang (1993) uses stochastic supply, our approach of conjecturing a price function and confirming it via market clearing is similar to his. Kyle, Obizhaeva, and Y. Wang (2017) use an agree-to-disagree approach, and the price function they obtain is very similar to ours, in the sense that it contains the public filters of the unobservable state variables rather than those variables themselves.

We differ from both papers in three respects. First, our information structure is heterogeneous but not hierarchical: investors cannot be ranked by how much they know, as they can in J. Wang (1993), where informed investors observe everything the uninformed observe. Second, our price is not fully revealing, whereas in Kyle, Obizhaeva, and Y. Wang (2017) it reveals a sufficient statistic for all private signals, so that each investor learns the others' information exactly. Third, we allow for general risk-averse preferences, while both papers assume constant absolute risk aversion. It is the combination of these three features in a dynamic setting that makes equilibrium difficult to characterize, and that our information structure is designed to deliver.

The most related paper to our work is Avdis and Glebkin (2025), which introduces the *CHILE* framework, a static model with accumulating noise in agents' private signals represented by Brownian increments. Their analysis demonstrates how such an information structure can generate a well-defined equilibrium without exogenous noise, allowing for the study of how wealth inequality affects market quality. Inspired by this idea, our paper develops a model to analyze trade persistence in competitive markets, where the accumulating noise evolves as a Brownian sheet, the two-parameter analogue of Brownian motion. From a technical perspective, this model is more involved, as it requires working with stochastic integration and equilibrium conditions in two dimensions (over both agents and time). Economically, the random-field formulation allows us to study the joint evolution of information and market variables, enabling the analysis of dynamic quantities such as trading activity, price informativeness, and wealth aggregation.

Outline. The remainder of this paper is organized as follows. In Section 2 we formulate the model, and we describe the discretization of the continuous economy. In Section 3 we derive the continuous economy as the limit of the discrete economies by solving the investors' utility maximization problems and aggregating the optimal individual demands, and we provide conditions for a trade equilibrium to exist. In Section 4 we discuss general properties of our market, including price informativeness and trading activity. In Section 5 we study price informativeness and wealth effects, and in particular how changes in the wealth allocation affect the market quality measures introduced in Section 4. In Section 6 we conclude our findings. For better readability, proofs and technicalities are deferred to the appendix.

2. Setup

The main object of interest is a financial market with a continuum of investors. All of our analysis is based on this *continuous economy.* The continuous economy is derived as the limit of a sequence of *discrete economies*, each of which consists of finitely many agents, and we therefore also refer to it as the *limiting economy.* In this section we first set up the continuous economy, and then we characterize the discrete economy as a discretization of our setup.

2.1. The continuous economy

In the following we characterize the asset structure, the information dynamics and the preferences.

2.1.1. *Asset structure.* We consider a continuous-time financial market consisting of two assets, a risk-free asset with price $B = (B_t)_{t\geq 0}$ and a risky asset with price $S = (S_t)_{t\geq 0}$ and constant supply θ. The risk-free asset satisfies $\mathrm{d}B_t = rB_t\,\mathrm{d}t$ for a constant interest rate $r > 0$. The risky asset pays dividends $D = (D_t)_{t\geq 0}$ whose dynamics are given by the stochastic differential equation

$$\mathrm{d}D_t = \alpha_D(G_t - D_t)\,\mathrm{d}t + \sigma_D\,\mathrm{d}W_t^D, \tag{2.1}$$

where α_D and σ_D are positive constants, and W^D is a Brownian motion. The short-term mean $G = (G_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$\mathrm{d}G_t = \alpha_G(g - G_t)\,\mathrm{d}t + \sigma_G\,\mathrm{d}W_t^G, \tag{2.2}$$

where α_G and σ_G are positive constants, g can have either sign, and W^G is a Brownian motion independent of W^D. Prices and dividends are publicly observed, whereas G is not.

2.1.2. *Information dynamics.* There is a continuum of agents located on the interval $[0, 1)$. Private information is a two-parameter stochastic process $\xi = (\xi_{(a,t)})_{(a,t)\in[0,1)\times\mathbb{R}_+}$ that evolves in agent-time space $[0, 1) \times \mathbb{R}_+$. Since there is a continuum of investors, each one of them is infinitesimally small and thus receives an infinitesimal amount of private information. In particular, investor $a \in [0, 1)$, who is located on the interval $[a, a + \mathrm{d}a) \subseteq [0, 1)$, observes the private signal $\mathrm{d}\xi_{(a,t)}$, with law of motion over a and t

$$\mathrm{d}\xi_{(a,t)} = \nu(a)\,\mathrm{d}G_t\,\mathrm{d}a + \mathrm{d}W^\xi_{(a,t)}. \tag{2.3}$$

In (2.3) the differentials $\mathrm{d}\xi_{(a,t)}$ and $\mathrm{d}W^\xi_{(a,t)}$ are two-dimensional: it is the increment of ξ over the infinitesimal rectangle $[a, a + \mathrm{d}a) \times [t, t + \mathrm{d}t)$. By contrast, $\mathrm{d}G_t$ is a one-dimensional differential. The function $\nu\colon [0, 1) \to (0,\infty)$ describes the precision of agents' private information. We assume that ν is measurable and Riemann integrable. The random field $W^\xi = (W^\xi_{(a,t)})_{(a,t)\in[0,1)\times\mathbb{R}_+}$ is a Brownian sheet, which can be seen as the two-parameter analogue of Brownian motion (see, e.g., Adler and Taylor (2007)), and which describes the noise in private signals. Appendix A defines the Brownian sheet, contrasts it with Brownian motion, and collects the properties that we use; in particular, fixing either parameter leaves a Brownian motion after rescaling (Corollary A.1). It is assumed that the Brownian sheet W^ξ is independent of both Brownian motions W^D and W^G.

2.1.3. *Public information.* Public information is described by a weighted average of the private signals (2.3). Let $w\colon [0,1) \to \mathbb{R}$ be a weight function, not known a priori but determined in equilibrium via market clearing, and let

$$\Xi_t := \int_{[(0,0),(1,t))} w(a)\,\mathrm{d}\xi_{(a,s)} \tag{2.4}$$

denote the resulting *public signal.* In particular, the stochastic integral in (2.4) runs over the rectangle $[0,1) \times [0,t)$, so that it accumulates over time as well as across agents. Integrating the private-signal differential (2.3) across the agent space against w gives

$$\mathrm{d}\Xi_t = \int_0^1 w(a)\,\mathrm{d}\xi_{(a,t)} = \left(\int_0^1 \nu(a)w(a)\,\mathrm{d}a\right)\mathrm{d}G_t + \int_0^1 w(a)\,\mathrm{d}W^\xi_{(a,t)}, \tag{2.5}$$

where on the right the integral runs over the agent coordinate a at fixed t. Aggregation across agents thus leaves a one-parameter process, and, as we will show shortly, the cross-sectional structure of the private signals enters it through the two constants

$$\mu := \int_0^1 \nu(a)w(a)\,\mathrm{d}a, \qquad \sigma_\xi^2 := \int_0^1 w(a)^2\,\mathrm{d}a.$$

By Proposition A.3 the aggregated noise is a Brownian motion scaled by σ_ξ, so that

$$W^\Xi_t := \frac{1}{\sigma_\xi}\int_{[(0,0),(1,t))} w(a)\,\mathrm{d}W^\xi_{(a,s)}$$

is a standard Brownian motion, and (2.5) reads

$$\mathrm{d}\Xi_t = \mu\,\mathrm{d}G_t + \sigma_\xi\,\mathrm{d}W^\Xi_t. \tag{2.6}$$

The public signal is therefore a classical one-parameter Itô process, and the cross-sectional structure of the private signals enters through μ and σ_ξ.

Although $\mu = 0$ is a potential solution, we exclude this case because the public signal would then carry no information. This is immediate from the stochastic differential (2.6), which reduces to $\mathrm{d}\Xi_t = \sigma_\xi\,\mathrm{d}W^\Xi_t$ when $\mu = 0$: the public signal no longer loads on the fundamental at all, and is a Brownian motion independent of G.

2.1.4. *Linear conjectured price structure.* In order to have a consistent information structure, the market clearing price S must exactly reveal the public signal Ξ. More precisely, the informational content of the price of the risky asset together with the dividends must be equivalent to the informational content of the public signal and the dividends. This information equivalence is achieved via the price conjecture

$$S_t = s_0 + s_D D_t + s_G \widehat{G}_t, \qquad \widehat{G}_t := \mathrm{E}[G_t | \mathcal{G}_t]. \tag{2.7}$$

The constants s_0, s_D and s_G are determined via market clearing. The σ-algebra $\mathcal{G}_t$ reflects public information. In other words, the set $\mathcal{G}_t$ contains all the information that is observable by all market participants at time t. Hence, we set $\mathcal{G}_t := \sigma(D_s, \Xi_s;\, s \le t)$, and we observe that $\mathcal{G}_t = \sigma(D_s, S_s;\, s \le t)$, which is an immediate consequence of the linear price structure. We use the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \ge 0}$ to describe the flow of public information.

2.1.5. *Preferences.* We consider a generalized utility function $U\colon [0,1) \times \mathbb{R} \to \mathbb{R}$ that describes the preferences of investors. In particular, for each fixed $a \in [0,1)$, the utility function $U(a,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is assumed to be twice continuously differentiable, strictly concave and strictly increasing, with coefficient of absolute risk aversion given by

$$\psi(a,\cdot) := -\frac{U''(a,\cdot)}{U'(a,\cdot)}.$$

We refer to $1/\psi(a,x)$ as investor a's absolute risk tolerance at wealth level x. This quantity will be the key channel through which wealth affects equilibrium holdings and trading activity.[5]

2.2. The discrete economy

To obtain a well-defined utility maximization problem for each investor, we discretize the previously defined continuous economy to obtain a discrete economy with finitely many investors. For this purpose, we use a sequence of partitions $\Pi = (\Pi_n)_{n \ge 2}$, where each element $\Pi_n = (a_i)_{0 \le i \le n-1}$ is a partition of the interval $[0,1)$, and where $|\Pi_n| := \max_{0 \le i \le n-1} |\Delta a_i| \to 0$ as $n \to \infty$. Here, we use the notation $\Delta a_i := a_{i+1} - a_i$, and we assume from now on that $\Delta a_i = \frac{1}{n}$. Moreover, without loss of generality, we assume that $a_i < a_{i+1}$. The economy with n agents, represented by the partition Π_n, is called the *n-th discrete economy.*

2.2.1. *Discrete information dynamics.* The discretization of the continuous economy is mainly concerned with the information structure. We replace the precision function ν and the weight

[5] Under decreasing absolute risk aversion (DARA), higher wealth corresponds to higher absolute risk tolerance.

function w by the simple functions

$$\nu^n(a) := \sum_{i=0}^{n-1} \nu(a_i)\mathbb{1}_{[a_i,a_{i+1})}(a), \qquad w^n(a) := \sum_{i=0}^{n-1} w(a_i)\mathbb{1}_{[a_i,a_{i+1})}(a),$$

and write ξ^n for the private-signal field obtained from (2.3) with ν replaced by ν^n, that is $\mathrm{d}\xi^n_{(a,t)} = \nu^n(a)\,\mathrm{d}G_t\,\mathrm{d}a + \mathrm{d}W^\xi_{(a,t)}$. In the n-th discrete economy investor a_i then observes the private signal increment

$$\mathrm{d}\big(\Delta_1\xi^n_{(a_i,t)}\big) = \nu(a_i)\,\mathrm{d}G_t\Delta a_i + \mathrm{d}\big(\Delta_1 W^\xi_{(a_i,t)}\big), \tag{2.8}$$

where the subscript in Δ_1 records that the increment is taken with respect to the first argument, that is across agents ($\Delta_1\xi^n_{(a_i,t)} := \xi^n_{(a_{i+1},t)} - \xi^n_{(a_i,t)}$, and likewise for the Brownian sheet $\Delta_1 W^\xi_{(a_i,t)} := W^\xi_{(a_{i+1},t)} - W^\xi_{(a_i,t)}$), over the interval $[a_i, a_{i+1})$ that investor a_i occupies. For fixed i these are one-parameter processes in t, so $\mathrm{d}\big(\Delta_1\xi^n_{(a_i,t)}\big)$ is their ordinary differential. Since ν^n is constant on $[a_i, a_{i+1})$, the drift in (2.8) is exactly $\nu(a_i)\Delta a_i$; this is why the discretization acts on ν rather than on ξ directly. Equation (2.8) is the discrete counterpart of (2.3), with the continuous differential $\mathrm{d}\xi_{(a,t)}$ replaced by its discrete analogue.

Setting $\Xi^n_t := \int_{[(0,0),(1,t))} w^n(a)\,\mathrm{d}\xi^n_{(a,s)}$ for the resulting public signal and

$$\mu_n := \int_0^1 \nu^n(a)w^n(a)\,\mathrm{d}a, \qquad \sigma_{\xi,n} := \left(\int_0^1 w^n(a)^2\,\mathrm{d}a\right)^{1/2}$$

for the two constants it produces, the argument that led to (2.6) shows that Ξ^n evolves as

$$\mathrm{d}\Xi^n_t = \mu_n\,\mathrm{d}G_t + \sigma_{\xi,n}\,\mathrm{d}W^{\Xi,n}_t, \tag{2.9}$$

where $W^{\Xi,n}_t := \sigma^{-1}_{\xi,n}\int_{[(0,0),(1,t))} w^n(a)\,\mathrm{d}W^\xi_{(a,s)}$ is a standard Brownian motion by Proposition A.3. The discrete public signal is thus of the same form as its continuous counterpart (2.6), with μ and σ_ξ replaced by μ_n and $\sigma_{\xi,n}$.

2.2.2. *Discrete price dynamics.* We discretize the price conjecture (2.7) by substituting the information set $\mathbb{G}$ with $\mathbb{G}^n = (\mathcal{G}^n_t)_{t\geq 0}$, where we denote $\mathcal{G}^n_t = \sigma(D_s, \Xi^n_s;\ s \leq t)$. In addition, we add a risk premium of the form $\rho\frac{1}{n}$ to the price in (2.7). In particular, we use the price conjecture

$$S^n_t = s_0 + \frac{\rho}{n} + s_D D_t + s_G\widehat{G}^n_t, \qquad \widehat{G}^n_t := \mathrm{E}[G_t|\mathcal{G}^n_t]. \tag{2.10}$$

Note well that the constants s_0, s_D and s_G will be determined via market clearing, which we carry out only in the continuous economy. The risk premium ρ will also be determined via market clearing.

The risk premium enters the discrete price, but not the continuous one, for a very specific reason. The aggregate supply θ of the risky asset is finite and has to be held in equilibrium. In the n-th discrete economy it is borne by n investors, so each carries risk of order $1/n$ and requires compensation of the same order to be willing to hold it; a price without a term of order $1/n$ would leave investors unwilling to absorb the supply, and the market would not clear. As n grows, the same finite risk is divided among ever more investors, the compensation each of them requires vanishes, and the premium disappears from the price: $\rho/n \to 0$, which is why (2.7) carries no such term. The constant ρ itself does not vanish. It is pinned down by market clearing in the limit and appears in the equilibrium demand of Section 3, where it is computed in Proposition F.4.

Since $\widehat{G}_t^n \to \widehat{G}_t$ almost surely as $n \to \infty$ (see Lemma C.7 together with Corollary C.8), the discrete price S^n converges to the continuous price S.

2.2.3. *Agree to disagree.* In the continuous economy private signals are merely an infinitesimal increment and thus negligible. Consequently, in the continuous economy there is no difference between the information set that is generated by public and private signals, and the information set that is generated by public signals only. Conversely, in each discrete economy the information set generated by public and private signals is strictly greater than the public information set, in general. More precisely, in the n-th discrete economy we may define the filtration $\mathbb{G}^{a_i,n} = (\mathcal{G}_t^{a_i,n})_{t\geq 0}$ via $\mathcal{G}_t^{a_i,n} = \sigma(D_s, \Xi_s^n, \Delta_1\xi_{(a_i,s)};\ s \leq t)$, which reflects all available information of investor a_i, and which in general satisfies $\mathcal{G}_t^n \subsetneq \mathcal{G}_t^{a_i,n}$. As a result, in addition to the public filter $\widehat{G}_t^n = \mathrm{E}[G_t|\mathcal{G}_t^n]$, we may also define the private filter $\widehat{G}_t^{a_i,n} = \mathrm{E}[G_t|\mathcal{G}_t^{a_i,n}]$, which eventually leads to the *inference wedge* $\Lambda^{a_i,n} = (\Lambda_t^{a_i,n})_{t\geq 0}$ defined via $\Lambda_t^{a_i,n} := \widehat{G}_t^{a_i,n} - \widehat{G}_t^n$. The inference wedge follows Ornstein-Uhlenbeck-type dynamics adapted to the filtration $\mathbb{G}^{a_i,n}$ with long-term mean zero and steady state mean reversion parameter $\alpha_\Lambda^n = \lim_{t\to\infty} \alpha_\Lambda^n(t)$ (see Corollary C.3 for the details).

In the discrete economies we now let investors *agree to disagree* about the extent to which their inference wedge is reflected in the market clearing price. To state this precisely it is convenient to work with the *instantaneous excess dollar return*

$$\mathrm{d}R_t^n = (D_t - rS_t^n)\,\mathrm{d}t + \mathrm{d}S_t^n, \tag{2.11}$$

where r is the net interest rate. Substituting the price conjecture (2.10) and writing the result in investor a_i's filtration $\mathbb{G}^{a_i,n}$, the drift of R^n loads on three state variables: the dividend D_t, the investor's own inference $\widehat{G}_t^{a_i,n}$, and the inference wedge $\Lambda_t^{a_i,n}$. The first two are already present in the continuous economy; the wedge is not, and it is the only one investors disagree about. Under the objective measure P the excess return therefore takes the form

$$\mathrm{d}R_t^n = \beta_n^* \Lambda_t^{a_i,n}\,\mathrm{d}t + (\text{other drift terms}) + (\text{diffusion terms}), \tag{2.12}$$

where the other drift terms load on 1, D_t and $\widehat{G}_t^{a_i,n}$ only, and where the objective loading on the wedge is

$$\beta_n^* := s_G(r + \alpha_\Lambda^n).$$

It is common to all investors, and is pinned down by the price coefficient s_G together with the speed α_Λ^n at which the wedge mean-reverts. The remaining coefficients play no role in the argument that follows; they are recorded in Appendix B.

Disagreement is introduced by individualizing this single loading. Investor a_i evaluates the economy under an equivalent probability measure $P^{a_i,n} \approx P$ that leaves the diffusion terms and the loadings on 1, D_t and $\widehat{G}_t^{a_i,n}$ untouched, and replaces β_n^* by an investor-specific constant $\beta(a_i)$, so that

$$\mathrm{d}R_t^n = \beta(a_i)\Lambda_t^{a_i,n}\,\mathrm{d}t + (\text{the same other drift terms}) + (\text{the same diffusion terms}). \tag{2.13}$$

The two drifts differ only by $(\beta(a_i)-\beta_n^*)\Lambda_t^{a_i,n}$, so passing from P to $P^{a_i,n}$ is a classical Girsanov transformation; the density process and the conditions under which it is well defined are given in Appendix D. The single number $\beta(a_i)$ thus summarizes investor a_i's belief. Agreement corresponds to $\beta(a_i) = \beta_n^*$, in which case no measure change takes place at all, while $\beta(a_i) = 0$ is the opposite extreme, in which investor a_i prices the asset as if the price contained their own private filter rather than the public one.

Two features of this construction are worth emphasizing. First, beliefs are individualized through the loading on the wedge and through nothing else: investors disagree about how private information is impounded in the price, but not about dividends, about their own inference, or about risk. Second, the object they disagree about is itself vanishing. The inference wedge satisfies $\Lambda_t^{a_i,n} \to 0$ as $n \to \infty$ by Corollary C.8, so that the drift distortion $(\beta(a_i) - \beta_n^*)\Lambda_t^{a_i,n}$ vanishes with it and eventually $P^{a_i,n} \to P$. Investors are in this sense *near-rational*: in any finite economy they may hold materially different views, but the measures under which they optimize are correct in the limit. Individual disagreement therefore disappears in the continuous economy, while its aggregate consequences do not, and it is those aggregate consequences that the rest of the paper develops.

To talk about the belief structure of the entire market, we define the *belief function* $\beta^n\colon [0,1) \to \mathbb{R}$ of the n-th discrete economy as the simple function

$$\beta^n(a) = \sum_{i=0}^{n-1} \beta(a_i)\mathbb{1}_{[a_i,a_{i+1})}(a).$$

The function $\beta\colon [0,1) \to \mathbb{R}$ is the belief function of the continuous economy. We assume that β is measurable and Riemann integrable.

2.2.4. *Wealth dynamics and admissibility.* For each investor a_i in the n-th discrete economy, trading strategies are modeled by $\mathbb{G}^{a_i,n}$-adapted processes $H^{a_i,n} = (H_t^{a_i,n})_{t\geq 0}$ that are integrable with respect to S^n. Consumption processes $c^{a_i,n} = (c_t^{a_i,n})_{t\geq 0}$ are $\mathbb{G}^{a_i,n}$-adapted processes. The wealth dynamics of investor a_i are then given by

$$\mathrm{d}V_t^{a_i,n} = rV_t^{a_i,n}\,\mathrm{d}t + H_t^{a_i,n}\,\mathrm{d}R_t^n - c_t^{a_i,n}\,\mathrm{d}t \tag{2.14}$$

where R^n is the instantaneous excess dollar return defined in (2.11). Each investor has initial endowment $v_0(a_i)$, where $v_0\colon [0,1) \to \mathbb{R}_+$ is a deterministic function. We then say that $(c^{a_i,n}, H^{a_i,n})$ is $v_0(a_i)$-*admissible*, if

$$V_t^{a_i,n} = v_0(a_i) + r\int_0^t V_s^{a_i,n}\,\mathrm{d}s + \int_0^t H_s^{a_i,n}\,\mathrm{d}R_s^n - \int_0^t c_s^{a_i,n}\,\mathrm{d}s \geq 0 \qquad \forall t \geq 0.$$

We denote the set of $v_0(a_i)$-admissible trading strategy-consumption pairs by $\mathcal{A}(v_0(a_i))$.

2.2.5. *Utility maximization.* Each investor solves the utility maximization problem

$$\operatorname*{ess\,sup}_{(c^{a_i,n},H^{a_i,n})\in\mathcal{A}(v_0(a_i))} \mathrm{E}_{P^{a_i,n}}\left[\int_t^\infty \mathrm{e}^{-r(s-t)}\,U(a_i, c_s^{a_i,n})\,\mathrm{d}s \,\middle|\, \mathcal{G}_t^{a_i,n}\right]. \tag{2.15}$$

In particular, in the n-th discrete economy each investor maximizes the expected discounted utility of future consumption conditional on all information available to them, and under the subjective measure $P^{a_i,n}$ pinned down by their belief $\beta(a_i)$ in (2.13).

Remark 2.1. A key modeling choice is the use of a Brownian sheet rather than a sequence of independent Brownian motions across agents. In an economy with a continuum of investors, independent noise typically vanishes under aggregation, which eventually leads to a no-trade equilibrium. To illustrate this fact, assume that we replace the Brownian sheet W^ξ in the private signals with a sequence $(W^a)_{0\leq a\leq 1}$ of independent Brownian motions. In particular, assume that equation (2.3) is replaced by $\mathrm{d}\xi_t^a = \nu(a)\,\mathrm{d}G_t\,\mathrm{d}a + \mathrm{d}W_t^a$. In the n-th discrete economy, the public signal would then be of the form

$$\sum_{i=0}^{n-1} w(a_i)(\xi_t^{a_i} - \xi_0^{a_i})\Delta a_i = \left(\sum_{i=0}^{n-1} w(a_i)\nu(a_i)\Delta a_i\right)(G_t - G_0) + \sum_{i=0}^{n-1} w(a_i)W_t^{a_i}\Delta a_i.$$

By using a version of the strong law of large numbers (see, e.g., Theorem 7.2 in Williams (1991)), aggregate noise vanishes as $n \to \infty$, yielding a fully revealing price. Thus, a continuum of independent Brownian motions cannot generate persistent aggregate noise.

Another important reason for choosing a Brownian sheet instead of a sequence of independent Brownian motions is the following observation. The noise terms in the private signals are the only stochastic variables in the entire financial market that are individually tied to the investors. By using a Brownian sheet, this individual stochasticity survives the passage from the discrete to the continuous economy. In contrast, by working with a sequence of independent Brownian motions, this individual stochastic factor is lost when passing to the continuous economy. Hence, working with a sequence of independent Brownian motions immediately implies: (i) all agents must choose identical (or zero) risky positions; (ii) disagreement cannot survive aggregation; and (iii) equilibrium generically collapses to no trade. This is the dynamic version of the Milgrom-Stokey logic. As a result, a no-trade equilibrium always exists, but non-trivial equilibria generically do not exist, because aggregate stochastic terms cannot be balanced without forcing zero positions almost everywhere.

3. The continuous economy as a limiting discrete economy

In this section we derive the continuous economy as a particular limit of the discrete economy. We do this by aggregating individual demands, derived in the discrete economy, and by taking the limit as $n \to \infty$. We observe that as individual demands converge to zero, aggregate demand accumulates and converges to something that is non-zero in general.

3.1. Cumulative demand and market clearing

For wealth x and wedge y, let $J^{a_i,n}(t,x,y)$ be the time-t value function that solves the HJB equation associated with problem (2.15). Let $\beta^n\colon [0,1) \to \mathbb{R}$ be the belief function of the n-th discrete economy. As we show in Appendix E under certain regularity conditions, the optimal trading strategy is

$$
\begin{aligned}
(H_t^{a_i,n})^* = &-\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}\cdot\frac{\beta^n(a_i)}{\eta_{VV}^{a_i,n}}\cdot\Lambda_t^{a_i,n} \\
&+\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}\cdot\frac{r\rho}{\eta_{VV}^{a_i,n}}\cdot\frac{1}{n}-\frac{J_{xy}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}\cdot\frac{\eta_{V\Lambda}^{a_i,n}}{\eta_{VV}^{a_i,n}},
\end{aligned}
$$

where $\eta_{VV}^{a_i,n}$ and $\eta_{V\Lambda}^{a_i,n}$ are constants corresponding to the quadratic (co-)variation of processes $V^{a_i,n}$ and $\Lambda^{a_i,n}$.[6] In line with standard results in portfolio optimization, the optimal holdings of each agent have three components: a state-dependent component reflecting the conditional expected return, a constant component reflecting the unconditional instantaneous myopic return, and a component reflecting hedging concerns between state variables (in this case, wealth V and wedge Λ).

To better understand the optimal trading strategy in the continuous economy, let us analyze its mean and variance at the large-n limit. As we prove in the appendix (see Proposition F.1), under the additional assumption

$$
\lim_{n\to\infty}\sqrt{n}\left\|\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}-\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}\right\|_{L^2(P)}=0 \tag{3.1}
$$

[6]See equation (E.1) of the Appendix for details. The risk premium ρ is given in Proposition F.4.

about the convergence speed of $\frac{J_x^{a_i,n}}{J_{xx}^{a_i,n}}$, it holds that

$$\lim_{n\to\infty} n\,\mathrm{E}[(H_t^{a_i,n})^*] = \theta \frac{\frac{1}{\psi(a_i, rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(u, rv_0(u))}\,\mathrm{d}u}.$$

Hence, as commonly seen in competitive Rational Expectations Equilibria (REE), average risky holdings are allocated according to relative risk tolerance. More precisely, each trader's average position is proportional to their absolute risk tolerance relative to aggregate risk tolerance in the economy. Unlike in most REE models with CARA preferences, however, absolute risk tolerance may depend on the trader's wealth. Wealth therefore affects equilibrium holdings through its effect on risk aversion. For example, under DARA, wealthier agents have higher absolute risk tolerance and therefore hold a larger average share of the risky asset. Under CARA, by contrast, this wealth channel is absent, and average holdings do not depend on wealth.

We also find that in steady state ($t \to \infty$) the variance of agent holdings has the limit

$$\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \frac{w^2(a_i)}{\int_0^1 w^2(u)du} \cdot \frac{\mathcal{V}^2_{(1,t)}}{2\alpha_\Lambda},$$

where $\mathcal{V}^2_{(1,t)}$ is total trading activity (see Section 4.2 below).

As we can see, holdings with higher variance correspond to private signals with a larger weight in the public signal. This means that agents who trade more aggressively affect the public signal more, as their position is conveyed with more weight. Moreover, whenever holdings become more variable, the total volume of trade increases. At the same time, holdings become less variable as the speed of mean reversion of the inference wedge increases, because the inference wedge itself becomes less variable.

3.1.1. *The cumulative demand function.* If we assume that the leading order partial differential equation that describes $\mathrm{e}^{-rt}\,J^{a_i,n}$ is of the form

$$-\tilde{J}_t^{a_i,n} = rx\tilde{J}_x^{a_i,n} + \mathrm{e}^{-rt}\,U(a_i, U'(a_i,\cdot)^{-1}(\mathrm{e}^{rt}\,\tilde{J}_x^{a_i,n})) - U'(a_i,\cdot)^{-1}(\mathrm{e}^{rt}\,\tilde{J}_x^{a_i,n})\tilde{J}_x^{a_i,n},$$

whose solution is given by $\tilde{J}^{a_i,n}(t,x) = \mathrm{e}^{-rt}\,\frac{1}{r}U(a_i, rx)$, we can show that the cumulative demand

$$X^n_{(a,t)} \coloneqq \sum_{\substack{0\le i\le n-1,\\ a_i<a}} (H_t^{a_i,n})^*, \qquad (a,t)\in[0,1)\times\mathbb{R}_+, \tag{3.2}$$

converges to the process $X = (X_{(a,t)})_{(a,t)\in[0,1)\times\mathbb{R}_+}$, which satisfies the identity

$$\begin{aligned} X_{(a,t)} = &-\frac{\rho\,(1+\alpha_\Lambda t)}{\eta_{VV}}\int_0^a \frac{1}{\psi(u,rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(a,s)}\,\mathrm{d}s \\ &+ \int_0^a \frac{\beta(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2}{r\eta_{VV}\psi(u,rv_0(u))}\,\mathrm{d}u \int_0^t \mathrm{d}f_1\big(D_s,\widehat{G}_s,\Xi_s\big) \\ &+ \int_{[(0,0),(a,t))} \frac{\beta(u)(\nu(u)\sigma_\xi^2-w(u)\mu)}{r\eta_{VV}\psi(u,rv_0(u))}\,\mathrm{d}f_2\big(\xi_{(u,s)},\widehat{G}_s,\Xi_s\big), \end{aligned} \tag{3.3}$$

where f_1 and f_2 are given in Theorem C.9. The proof of the market clearing equation (3.3) is given in Appendix F (see Proposition F.2). It builds on several technical assumptions and results about the value functions $J^{a_i,n}$, which are discussed in full detail in Appendix E. One particular assumption that has to be additionally imposed in Proposition F.2 for the market clearing equation (3.3) to hold in this exact form is that the convergence of the ratios must satisfy

$$\lim_{n\to\infty}\frac{1}{n}\sum_{0\le i\le n-1}\sqrt{n}\left\|\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}-\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}\right\|_{L^2(P)}=0, \tag{3.4}$$

or an equivalent condition that ensures a certain convergence speed. It is worth comparing this condition to (3.1) which is similar, but on an individual level. We emphasize that the additional assumption (3.4) is not a necessary condition to study the convergence of the discrete market clearing equation (3.2). However, without an assumption of this form, the market clearing equation (3.3) might be different, which especially means that additional terms could appear in (3.3). In particular, a condition of the form as in (3.4) ensures that all the covariance terms

$$\mathrm{Cov}\left(\left(\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}-\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}\right)\left(\frac{J_x^{a_j,n}(t,V_t^{a_j,n},\Lambda_t^{a_j,n})}{J_{xx}^{a_j,n}(t,V_t^{a_j,n},\Lambda_t^{a_j,n})}-\frac{J_x(t,v_0(a_j),0)}{J_{xx}(t,v_0(a_j),0)}\right),\Lambda_t^{a_i,n}\Lambda_t^{a_j,n}\right)$$

vanish in aggregation. Otherwise, the market clearing equation (3.3) would also contain mixed terms. We show in Appendix F, Lemma F.3, that $\mathrm{Cov}(\Lambda_t^{a_i,n},\Lambda_t^{a_j,n}) = \mathrm{E}[\Lambda_t^{a_i,n}\Lambda_t^{a_j,n}]$ is of order $\Delta a_i\Delta a_j$. However, if condition (3.4) (or an equivalent) does not hold, Lemma F.3 might not be enough to ensure that the market clearing equation has the exact form (3.3), but instead may contain additional terms. The economic intuition is that the difference of the ratios in (3.4) is understood as a first order Taylor approximation in agent-direction of the continuous function J_x/J_{xx}, i.e.,

$$\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}-\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}\approx\frac{\partial}{\partial x}\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}v_0'(a_i)\Delta a_i. \tag{3.5}$$

Here, the difference $J_x^{a_i,n}/J_{xx}^{a_i,n}-J_x/J_{xx}$ is no longer understood as the difference between the (ratios of) value functions in the discrete and continuous economy, but instead as a difference of interval length on which agent a_i lives. More precisely, in the continuous economy there is only the function J_x/J_{xx} which is the same for all investors. Agent-dependence enters via initial wealth, which is the function $v_0\colon [0,1) \to \mathbb{R}_+$. Since in the continuous economy a single agent is negligible—agent a lives on $[a, a+\mathrm{d}a)$—wealth remains unchanged as time evolves. Hence, from the perspective of an individual agent, wealth can only change over time, if the interval length has positive measure, i.e., if agent a lives on $[a, a+\Delta a)$ with $\Delta a > 0$. The change in the ratio J_x/J_{xx} caused by such an interval enlargement is what the approximation in (3.5) heuristically describes. Note, however, that (3.5) is only used to give some interpretation of condition (3.4). We neither need a third partial derivative J_{xxx}, nor do we require v_0 to be continuously differentiable.

As we can see in (3.3), the cumulative demand process X has three components. The first component, shown in the first line, represents deterministic risk sharing across heterogeneous agents.

The second component, shown in the second line, captures stochastic innovations driven by public information. These innovations are common to all agents, originating as shocks to dividends and to the price signal. The innovations are weighted by the coefficient term

$$\big(w(u)\mu - \nu(u)\sigma_\xi^2\big)^2 = \sigma_\xi^4\Big(w(u)\frac{\mu}{\sigma_\xi^2} - \nu(u)\Big)^2$$

which measures the mismatch between the price weight assigned to agent u's information and the precision-weighted informativeness of that agent's private signal. Here μ/σ_ξ^2 can be interpreted as the signal loading of the public signal per unit of noise variance, while $\nu(u)$ measures the local informativeness of agent u's private signal about the short-term dividend mean.

The third component, shown in the third line, captures stochastic innovations driven by the private information of individual agents. The innovation term $\mathrm{d}f_2\big(\xi_{(u,s)}, \widehat{G}_s, \Xi_s\big)$ combines agent u's own signal innovation

$$\mathrm{d}\xi_{(u,s)} - \nu(u)\alpha_G\big(g - \widehat{G}_s\big)\,\mathrm{d}s\,\mathrm{d}u$$

with the public innovation $\mathrm{d}\Xi_s - \mu\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s$, weighted by the filtering coefficients given in Theorem C.9. Netting out the public innovation leaves the part of agent u's private signal that is not explained by public information. This part is idiosyncratic in the agent dimension and is driven by the Brownian-sheet noise in private signals. It affects cumulative demand because agents with different disagreement parameters, different risk tolerances, and different signal precisions react differently to their private information.

Both stochastic components carry the same factor $1/\psi(u, rv_0(u))$ as the deterministic first component, namely agent u's absolute risk tolerance. The risk-tolerance channel that determines average holdings therefore also determines how strongly private information and disagreement feed into cumulative demand.

Moreover, both stochastic components are proportional to $\beta(u)$, which is where disagreement enters. The sign and magnitude of $\beta(a)$ summarize how agent a's subjective price dynamics differ from the common benchmark β_n^* of (2.12): a positive value means that the agent behaves as if the inference wedge raises expected returns, a negative value means the opposite, and agreement is the single value $\beta(a) = \beta_n^*$. Cross-sectional variation in β therefore creates opposing speculative demands, and this is what sustains trade even though the individual distortions vanish in the large-market limit.

3.1.2. *Market clearing condition.* Based on the cumulative demand function we can now define market clearing in the continuous economy. The *market clearing condition* (MC) in the continuous economy is given by

$$X_{(1,t)} \stackrel{!}{=} \theta, \qquad \theta \in \mathbb{R}, \qquad \forall t \geq 0. \tag{MC}$$

The interpretation of condition (MC) is that there is a fixed amount of stock, θ, at every time $t \geq 0$. Hence, cumulative demand $X_{(1,t)}$, which is obtained by integrating over the entire agent space $[0, 1)$, must be equal to θ at all times. For (MC) to hold true, i.e., for total aggregate demand to be constant and in particular deterministic, the stochastic terms in equation (3.3) must vanish when integrating over a from 0 to 1. We will use this fact as a condition to pin down the weight function w.

Assuming that such a weight function w has been found, we may also consider the "relaxed" market clearing condition (MC′), which reads as

$$X_{(1,t)} = -\frac{\rho\,(1+\alpha_\Lambda t)}{\eta_{VV}} \int_0^1 \frac{1}{\psi(u, rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(1,s)}\,\mathrm{d}s. \tag{MC′}$$

In fact, equation (MC′) is nothing other than aggregate demand $X_{(1,t)}$ from equation (3.3) with vanishing stochastic terms. We can now use condition (MC′) to pin down the risk premium ρ, so that the market eventually clears.

3.2. General equilibrium conditions

So far we have just assumed the existence of a weight function w, which reduces $X_{(1,t)}$ such that it satisfies (MC′). We now want to understand if such weight functions exist, and if so under which conditions. Since we can solve for the risk premium ρ for (MC′) to hold true, it is enough to work under the relaxed market clearing condition (MC′).

3.2.1. *Risk premium.* As we show in Proposition F.4 in the appendix, the risk premium ρ equals

$$\rho = -\frac{\eta_{VV}\theta}{\int_0^1 \frac{1}{\psi(u,rv_0(u))}du}.$$

This term represents a classical risk premium providing constant compensation to risk-averse agents for bearing risk and giving them an incentive to hold the risky asset.

3.2.2. *Trivial equilibria.* To satisfy the market clearing condition, the stochastic terms in (3.3) must vanish after aggregation over the agent space. There are two fundamentally different ways in which this can occur. First, the stochastic terms may vanish pointwise for almost every agent $a \in [0,1)$. In this case, cumulative demand is deterministic at every level, leading to a trivial equilibrium. Second, the stochastic terms may only vanish after aggregation over the entire agent space, while remaining non-zero at intermediate levels. In this case, individual demand is stochastic, but aggregate demand is deterministic, leading to a non-trivial equilibrium.

We first characterize the class of weight functions corresponding to trivial equilibria. We define an equilibrium as *trivial* at time $t \geq 0$, whenever

$$X_{(a,t)} = -\frac{\rho\,(1+\alpha_\Lambda t)}{\eta_{VV}} \int_0^a \frac{1}{\psi(u, rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(a,s)}\,\mathrm{d}s$$

for almost every $a \in [0,1)$. Although investors may trade, we call such equilibria trivial because there is no stochasticity in cumulative demand. In other words, investors only trade due to the deterministic risk premium ρ.

A key observation is that both stochastic integrals in (3.3) are driven by $\nu(a)\sigma_\xi^2 - w(a)\mu$. Eliminating this term pointwise removes all stochastic components, which leads to the following result.

Proposition 3.1. *A trivial equilibrium exists if and only if the weight function satisfies* $w(a) = \frac{\sigma_\xi^2}{\mu}\nu(a)$ *for almost every* $a \in \{\beta \neq 0\}$.

In fact, a trivial equilibrium always exists but is not necessarily unique. For instance, we can set $w(a) = \kappa\nu(a)$ for some constant $\kappa \neq 0$, so that

$$\sigma_\xi^2 = \int_0^1 w(a)^2\,\mathrm{d}a = \kappa^2 \int_0^1 \nu(a)^2\,\mathrm{d}a,$$
$$\mu = \int_0^1 \nu(a)w(a)\,\mathrm{d}a = \kappa \int_0^1 \nu(a)^2\,\mathrm{d}a = \frac{\sigma_\xi^2}{\kappa}.$$

This implies $\kappa = \frac{\sigma_\xi^2}{\mu}$ and therefore $w(a) = \frac{\sigma_\xi^2}{\mu}\nu(a)$ for all $a \in [0,1)$. Alternatively, we could set $w(a) = \kappa\nu(a)\mathbb{1}_{\{\beta\neq 0\}}(a)$ for some constant $\kappa \neq 0$, which implies

$$\sigma_\xi^2 = \int_0^1 w(a)^2 \,\mathrm{d}a = \kappa^2 \int_{\{\beta\neq 0\}} \nu(a)^2 \,\mathrm{d}a,$$
$$\mu = \int_0^1 \nu(a)w(a)\,\mathrm{d}a = \kappa \int_{\{\beta\neq 0\}} \nu(a)^2 \,\mathrm{d}a = \frac{\sigma_\xi^2}{\kappa}.$$

As before, this implies $\kappa = \frac{\sigma_\xi^2}{\mu}$ and thus $w(a) = \frac{\sigma_\xi^2}{\mu}\nu(a)$ for all $a \in \{\beta \neq 0\}$.

3.2.3. *Non-trivial equilibria.* We now turn to equilibria where stochastic fluctuations cancel only in the aggregate. In contrast to a trivial equilibrium, a *non-trivial equilibrium* at time $t \geq 0$ is characterized by the existence of a non-empty subset $A \subseteq [0,1)$, such that $X_{(a,t)}$ is non-deterministic with positive probability for all $a \in A$, i.e.,

$$\exists A \subseteq [0,1),\ A \neq \emptyset,\ \forall a \in A:$$
$$P\left[X_{(a,t)} \neq -\frac{\rho\,(1+\alpha_\Lambda t)}{\eta_{VV}} \int_0^a \frac{1}{\psi(u, rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(a,s)}\,\mathrm{d}s\right] > 0.$$

We have seen before that a trivial equilibrium always exists. In contrast, the existence of a non-trivial equilibrium is not guaranteed and heavily relies on the degree of disagreement, described by the function β. To understand this dependence we first derive the general structure of the weight function w for non-trivial equilibria.

More precisely, we seek a function $w\colon [0,1) \to \mathbb{R}$ such that the stochastic terms in the second and third lines of (3.3) vanish only after aggregation over the entire agent space. Equivalently, we require that $X_{(1,t)}$ satisfies the relaxed market clearing condition (MC′). At the same time, we require the existence of a subset of agents $A \subseteq [0,1)$ such that the stochastic terms in (3.3) do not vanish when integrating up to $a \in A$. Thus, while aggregate demand is deterministic, individual or partially aggregated demand remains stochastic. Economically, this corresponds to a situation in which market clearing holds in the aggregate, but some agents trade based on heterogeneous beliefs about the public signal and their private information. In this case, the stochastic integrals in (3.3) exhibit a bridge-like structure on $[0,1)$: they start and end at zero, but evolve stochastically in between.

To characterize the class of weight functions that give rise to such non-trivial equilibria, we rely on the following three observations. First, both stochastic integrals in (3.3) depend on the term $\frac{\beta(a)(\nu(a)\sigma_\xi^2 - \mu w(a))}{\psi(a, rv_0(a))}$. Second, the weight function satisfies the normalization conditions

$$\int_0^1 w(a)\big(\mu w(a) - \nu(a)\sigma_\xi^2\big)\,\mathrm{d}a = 0 \quad \text{and} \quad \int_0^1 w(a)\frac{\mu\sigma_G^2\nu(a) + w(a)}{\mu^2\sigma_G^2 + \sigma_\xi^2}\,\mathrm{d}a = 1,$$

which follow from $\int_0^1 w(a)^2\,\mathrm{d}a = \sigma_\xi^2$ and $\int_0^1 \nu(a)w(a)\,\mathrm{d}a = \mu$. Finally, the two-parameter stochastic integral $\int_{[(0,0),(1,t))} f(a)\, \mathrm{d}\xi_{(a,s)}$ can be rewritten as a double integral of the form

$$\int_{[(0,0),(1,t))} g(a) \bigg(\int_0^1 f(u)\, \mathrm{d}\xi_{(u,s)} \bigg)\, \mathrm{d}a \quad \text{for any function } g \text{ satisfying } \int_0^1 g(a)\, \mathrm{d}a = 1.$$

These observations allow us to characterize all weight functions that generate non-trivial equilibria, leading to the following result.

Proposition 3.2. *Under the general market clearing condition* (MC′) *the following two assertions are equivalent:*

1. *There exists a non-trivial equilibrium.*
2. *There exists a constant $\kappa \neq 0$, which may depend on μ and σ_ξ, such that the weight function $w\colon [0,1) \to \mathbb{R}$ satisfies*

$$w(a) = \begin{cases} \frac{\sigma_\xi^2 \nu(a)}{\mu + \kappa \frac{\psi(a, rv_0(a))}{\beta(a)}}, & a \in \{\beta \neq 0\}, \\ 0, & a \in \{\beta = 0\}, \end{cases}$$

is well-defined for a.e. $a \in [0,1)$, and is non-zero on a non-empty subset of $[0,1)$.

Remark 3.3. Note that Proposition 3.2 is a characterization of the weight function. Existence of non-trivial equilibria is addressed in the next section.

3.3. The existence of non-trivial equilibria

In the previous section, we have seen that trivial equilibria always exist, and we have described the general structure of weight functions in the presence of a non-trivial equilibrium. We now want to describe the conditions that must be satisfied in order for a non-trivial equilibrium to exist.

Proposition 3.4. *A non-trivial equilibrium exists if and only if $\{\beta \neq 0\}$ has positive measure and*

$$\int_{\{\beta \neq 0\}} \frac{\nu(u)^2}{\big(1 + \tilde{\kappa} \frac{\psi(u, rv_0(u))}{\beta(u)}\big)^2} \cdot \frac{\psi(u, rv_0(u))}{\beta(u)}\, \mathrm{d}u = 0 \tag{3.6}$$

for a constant $\tilde{\kappa} \neq 0$.

We note that for condition (3.6) to be satisfied, the belief function β must take on both positive and negative values; that is, the disagreement among different agents must be sufficiently dispersed. If all investors distort the price dynamics in the same direction, their demands move together and cannot generate the bridge-like cancellation required for market clearing. A trade equilibrium requires disagreement on both sides of the market:

some investors must behave as if the price underreacts to their information, while others must behave as if it overreacts.

The following result provides a sufficient condition for the existence of a non-trivial equilibrium, avoiding the auxiliary parameter $\tilde{\kappa}$.

Proposition 3.5. *Assume $\beta(\cdot)$ and $\psi(\cdot, rv_0(\cdot))$ are locally Lipschitz-continuous on $\{\beta \neq 0\}$ and β takes positive and negative values with $\inf_{\beta>0} \beta > 0 > \sup_{\beta<0} \beta$ and*

$$\int_{\{\beta\neq 0\}} \frac{\nu(u)^2 \psi(u, rv_0(u))}{\beta(u)} \, \mathrm{d}u \neq 0. \tag{3.7}$$

Then a non-trivial equilibrium exists.

Remark 3.6. It is worth noting that the two parameters κ and $\tilde{\kappa}$ appearing in Proposition 3.2 and Proposition 3.4, respectively, are related but are not the same object. They are connected via the identity $\kappa = \mu\tilde{\kappa}$. By dividing both sides of equation (3.6) by μ^2 we obtain

$$\int_{\{\beta\neq 0\}} \frac{\nu(u)^2}{\left(\mu + \kappa \frac{\psi(u, rv_0(u))}{\beta(u)}\right)^2} \cdot \frac{\psi(u, rv_0(u))}{\beta(u)} \, \mathrm{d}u = 0, \tag{3.8}$$

which is essentially the analogue of (3.6) in terms of κ. However, note that although equations (3.6) and (3.8) are very similar, the fixed point problem arising from equation (3.8) differs from the one given in Proposition 3.4. The reason is that μ is a function of κ since $\mu = \int_0^1 \nu(a) w(a) \, \mathrm{d}a$, and because κ appears in $w(a)$.

4. Key financial market measures

In the previous section we have seen how the continuous economy is derived as a limiting discrete economy, and how this convergence is related to the market clearing condition. We considered two types of equilibria: trivial equilibria, which always exist, and non-trivial equilibria, which only exist if the disagreement is heterogeneous enough. In both types of equilibria, market clearing ultimately pins down the risk premium. In this section we will derive further measures to analyze market quality.

4.1. Price informativeness

We start by analyzing the informational content of the price. For this purpose, we define the *price informativeness* as

$$\mathcal{I} = 1 - \lim_{t\to\infty} \frac{\mathrm{E}[\mathrm{Var}[G_t|\mathcal{G}_t]]}{\mathrm{Var}(G_t)}.$$

Note that this definition is a classical R^2-type measure, where $\mathrm{E}[\mathrm{Var}[G_t|\mathcal{G}_t]]$ represents the residual sum of squares and $\mathrm{Var}(G_t)$ is the total sum of squares. We take the limit because we are particularly interested in the steady state.

Proposition 4.1. *The price informativeness $\mathcal{I}$ satisfies*

$$\mathcal{I} = 1 - \frac{2\alpha_G \gamma}{\sigma_G^2} = \frac{\left(\sqrt{(\alpha_D^2\sigma_G^2 + \alpha_G^2\sigma_D^2)(\tau^2\sigma_G^2 + 1)} - \sigma_D\alpha_G\right)^2}{\tau^2\sigma_G^2(\alpha_D^2\sigma_G^2 + \alpha_G^2\sigma_D^2) + \alpha_D^2\sigma_G^2}, \qquad \tau^2 := \frac{\mu^2}{\sigma_\xi^2}.$$

4.1.1. *Dependence on the aggregate error.* From Proposition 4.1 we observe that price informativeness $\mathcal{I}$ essentially depends on three parameters: (1) the speed of mean-reversion of the process G, which is α_G, (2) the volatility of the process G, which is σ_G, and (3) the steady state mean squared error of the public filter $\widehat{G}$ in the continuous economy, which is γ. We may consider the first two components, i.e., α_G and σ_G, as fixed, since they are given by the model. The interesting component to understand price informativeness $\mathcal{I}$ is thus the mean squared error γ. In fact, we have

$$\frac{\partial \mathcal{I}}{\partial \gamma} = -\frac{2\alpha_G}{\sigma_G^2} < 0,$$

since $\alpha_G > 0$, which implies that price informativeness decreases with a higher mean squared error. This observation makes sense because a lower mean squared error corresponds to a better filter, which in turn means the price reveals a better estimate of the process G.

4.1.2. *Dependence on the precision of public information.* The steady state mean squared error γ satisfies

$$\gamma = \frac{\sigma_D\sqrt{(\alpha_D^2\sigma_G^2 + \alpha_G^2\sigma_D^2)(\tau^2\sigma_G^2 + 1)} - \alpha_G\sigma_D^2}{\tau^2(\alpha_D^2\sigma_G^2 + \alpha_G^2\sigma_D^2) + \alpha_D^2}, \qquad \tau^2 = \frac{\mu^2}{\sigma_\xi^2},$$

and thus changes in γ essentially are due to changes in the parameters μ and σ_ξ. As before, we may treat the constants α_D, α_G, σ_D and σ_G as fixed, since they are given by the model. A short calculation shows $\frac{\partial \gamma}{\partial \tau^2} \leq 0$. Hence, we find that the mean squared error is decreasing in τ^2. Since the public signal Ξ_t satisfies

$$\Xi_t = \mu(G_t - G_0) + \int_{[(0,0),(1,t))} w(u)\, \mathrm{d}W^\xi_{(u,s)},$$

and because the process

$$\frac{1}{\sigma_\xi}\int_{[(0,0),(1,t))} w(u)\, \mathrm{d}W^\xi_{(u,s)} = \frac{\int_0^1 w(u)\partial_1 W^\xi_{(u,t)}}{\left(\int_0^1 w(u)^2 du\right)^{1/2}}$$

is a Brownian motion, we may view τ^2 as the precision of the public signal.[7] By observing that

$$\frac{\partial \mathcal{I}}{\partial \tau^2} = \frac{\partial \mathcal{I}}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial \tau^2} \geq 0,$$

we see that price informativeness $\mathcal{I}$ increases as the precision τ^2 of the public signal increases. A higher precision in the public signal essentially means a better filter, thus a lower mean squared filtering error, and therefore more information to be obtained from the price. This is essentially the same conclusion as before.

4.2. Trading activity

The instantaneous trading activity $\mathcal{V}_{(a,t)}$ up to agent a at time t is defined as the square root of the partial derivative with respect to t of the quadratic variation of aggregate demand, i.e.,

$$\mathcal{V}_{(a,t)} := \left(\frac{\partial \langle X, X \rangle_{(a,t)}}{\partial t} \right)^{1/2} = \left(\frac{\partial}{\partial t} \int_{[(0,0),(a,t))} \mathrm{d}\langle X, X \rangle_{(u,s)} \right)^{1/2}.$$

The general idea here is to use the quadratic variation of aggregate demand to measure trading activity. Since we are interested in the instantaneous trading activity at time t, we take the partial derivative with respect to t because it reflects the change in variation of aggregate demand at that time.

Proposition 4.2. *The trading activity satisfies*

$$\mathcal{V}_{(a,t)} = \frac{\sigma_G^2 - \alpha_G \gamma}{\mu^2 \sigma_G^2 + \sigma_\xi^2} \left(\int_0^a \frac{\beta(u)^2 (\nu(u) \sigma_\xi^2 - w(u)\mu)^2}{r^2 \eta_{VV}^2 \psi(u, rv_0(u))^2} \, \mathrm{d}u \right)^{1/2}.$$

As we show in Appendix F (see equation (F.4)) the integral inside the square root in Proposition 4.2 satisfies

$$\int_0^a \frac{\beta(u)^2 (\nu(u) \sigma_\xi^2 - w(u)\mu)^2}{r^2 \eta_{VV}^2 \psi(u, rv_0(u))^2} \, \mathrm{d}u = \frac{\kappa^2}{r^2 \eta_{VV}^2} \int_0^a w(u)^2 \, \mathrm{d}u.$$

In particular, integrating from 0 to 1 yields that the total trading activity is

$$\mathcal{V}_{(1,t)} = \frac{\sigma_G^2 - \alpha_G \gamma}{\mu^2 \sigma_G^2 + \sigma_\xi^2} \cdot \frac{|\kappa|}{r \eta_{VV}} \left(\int_0^1 w(u)^2 \, \mathrm{d}u \right)^{1/2} = \frac{\sigma_G^2 - \alpha_G \gamma}{\mu^2 \sigma_G^2 + \sigma_\xi^2} \cdot \frac{|\kappa| \sigma_\xi}{r \eta_{VV}}. \tag{4.1}$$

Like price informativeness $\mathcal{I}$, trading activity $\mathcal{V}_{(1,t)}$ depends on μ and σ_ξ^2. Moreover, it depends on the absolute value of the constant $|\kappa|$, which is part of the definition of the weight function in Proposition 3.2. The interpretation of $|\kappa|$ is discussed in the next section.

[7] The partial derivative ∂_1 in $\partial_1 W^\xi_{(u,t)}$ indicates that the stochastic integral is only taken with respect to the first coordinate in $W^\xi_{(u,t)}$ (i.e., u), while the second coordinate is kept fixed.

Remark 4.3. By Proposition 3.1, the trading activity of a trivial equilibrium is zero.

4.3. Distance to a trivial equilibrium

We define the distance to a trivial equilibrium by the absolute value of the constant $\tilde{\kappa}$, as described in Proposition 3.4. In particular, we define $|\tilde{\kappa}|$ to be the distance to a trivial equilibrium, where $\tilde{\kappa} \in \mathbb{R}$ satisfies

$$\int_{\{\beta \neq 0\}} \frac{\nu(u)^2}{\left(1 + \tilde{\kappa}\frac{\psi(u, rv_0(u))}{\beta(u)}\right)^2} \cdot \tilde{\kappa}\frac{\psi(u, rv_0(u))}{\beta(u)} du = 0.$$

Note that the above equation is nothing other than condition (3.6) multiplied by $\tilde{\kappa}$. The reason for this adjustment is to include $\tilde{\kappa} = 0$, which essentially describes a trivial equilibrium.

Proposition 4.4. *Let w be an equilibrium weight function. The distance to a trivial equilibrium satisfies*

$$|\tilde{\kappa}| = \int_{\{\beta \neq 0\}} \left| \frac{w_{\text{trivial}}(u)}{w(u)} - 1 \right| \frac{|\beta(u)|}{\psi(u, rv_0(u))} du,$$

where $w_{\text{trivial}}(\cdot) = \frac{\sigma_\xi^2 \nu(\cdot)}{\mu} \mathbb{1}_{\{\beta \neq 0\}}(\cdot)$ is the minimal weight function of the trivial equilibrium.

As described in Remark 3.6 we generally have $\kappa = \tilde{\kappa}\mu$ (see also the proof of Proposition 3.4 given in Appendix F). In a non-trivial equilibrium, we further have $\mu = \sigma_\xi^2 > 0$. Hence, we find that the trading activity satisfies

$$\mathcal{V}_{(1,t)} = \frac{\sigma_G^2 - \alpha_G \gamma}{\mu^2 \sigma_G^2 + \sigma_\xi^2} \cdot \frac{|\kappa| \sigma_\xi}{r \eta_{VV}} = \frac{\sigma_G^2 - \alpha_G \gamma}{\mu \sigma_G^2 + 1} \cdot \frac{|\tilde{\kappa}| \sigma_\xi}{r \eta_{VV}}.$$

In other words, an increasing distance to a trivial equilibrium yields a higher trading activity. In a trivial equilibrium the trading activity is zero as seen before.

4.4. Level of disagreement

The level of disagreement is described by the L^2-norm of the function β. In particular, the level of disagreement is defined as

$$\|\beta\|_2 = \left(\int_0^1 \beta(u)^2 \,\mathrm{d}u \right)^{1/2}.$$

We can generally say that increasing the absolute value of β on a non-empty subset of $[0, 1)$ increases the level of disagreement. We further have the following result.

Proposition 4.5. *For a given belief function $\beta\colon [0,1) \to \mathbb{R}$, assume that a non-trivial equilibrium exists. Further, assume that its weight function w has distance $|\tilde{\kappa}|$. Then, for any constant $c \neq 0$ there exists a non-trivial equilibrium for the function $\beta' = c\beta$ with the same weight function w, whose distance to a trivial equilibrium is given by $|\tilde{\kappa}'| = |c\tilde{\kappa}|$.*

Proposition 4.5 implies that a scaled function $\beta' = c\beta$ results in a scaled trading activity $\mathcal{V}'_{(1,t)} = |c|\mathcal{V}_{(1,t)}$. Indeed, from Proposition 3.2 we know that the weight function is of the form $w(\cdot) = \frac{\sigma_\xi^2 \nu(\cdot)}{\mu\left(1+\tilde{\kappa}\frac{\psi(\cdot, rv_0(\cdot))}{\beta(\cdot)}\right)} \mathbb{1}_{\{\beta\neq 0\}}(\cdot)$, so that μ and σ_ξ^2 remain unchanged, and the statement follows from the identity $\mathcal{V}_{(1,t)} = \frac{\sigma_G^2 - \alpha_G \gamma}{\mu\sigma_G^2 + 1} \cdot \frac{|\tilde{\kappa}|\sigma_\xi}{r\eta_{VV}}$.

Corollary 4.6. *Consider a non-trivial equilibrium with belief function β, weight function w, distance to a trivial equilibrium $|\tilde{\kappa}|$, and absolute risk aversion ψ. A proportional increase in absolute risk aversion leads to a proportional decrease in the distance to a trivial equilibrium. In particular, keeping everything else fixed, for any $c > 0$, the equilibrium with absolute risk aversion $\psi' = c\psi$ has distance $|\tilde{\kappa}'| = \frac{1}{c}|\tilde{\kappa}|$ from a trivial equilibrium.*

Similarly, Corollary 4.6 implies that a scaled absolute risk aversion function $\psi' = c\psi$ results in an inversely scaled trading activity $\mathcal{V}'_{(1,t)} = \frac{1}{c}\mathcal{V}_{(1,t)}$. This comparative static should be interpreted as an aggregate risk-tolerance effect. A proportional increase in absolute risk aversion reduces the willingness of all agents to absorb risky positions, thereby moving the equilibrium closer to the trivial benchmark and lowering trading activity. When absolute risk aversion is wealth-dependent, changes in the wealth distribution can affect trading activity through the same risk-tolerance channel, although the effect then depends on how wealth is distributed across agents.

5. Market quality, wealth effects and the role of beliefs

5.1. Price efficiency and the no-trade situation

From the discussion in Section 4.1 we know that price informativeness satisfies $\mathcal{I} = 1 - \frac{2\alpha_G\gamma}{\sigma_G^2}$ (see Proposition 4.1), and we have seen that $\mathcal{I}$ depends on the weight function w only via aggregate precision $\tau^2 = \frac{\mu^2}{\sigma_\xi^2}$. In fact, we have shown in Section 4.1.2 that $\frac{\partial \mathcal{I}}{\partial \tau^2} \geq 0$.

In order to maximize price informativeness, we have to maximize aggregate precision τ^2. Here it is worth noting that

$$0 \leq \tau^2 = \frac{\left(\int_0^1 \nu(a)w(a)\,\mathrm{d}a\right)^2}{\int_0^1 w(a)^2\,\mathrm{d}a} \leq \frac{\int_0^1 \nu(a)^2\,\mathrm{d}a \int_0^1 w(a)^2\,\mathrm{d}a}{\int_0^1 w(a)^2\,\mathrm{d}a} = \int_0^1 \nu(a)^2\,\mathrm{d}a, \tag{5.1}$$

which follows from Hölder's inequality. From Proposition 3.1 we know that a trivial equilibrium always exists, and that its weight function is of the form $w(a) = \frac{\sigma_\xi^2}{\mu}\nu(a)$ for almost every $a \in \{\beta \neq 0\}$. Hence, we may simply set $w(a) = \nu(a)$, so that $\tau^2 = \int_0^1 \nu(a)^2 \, \mathrm{d}a$. In other words, price informativeness is maximal in the presence of a trivial equilibrium. Since investors do not trade in a trivial equilibrium, except for one initial trade when the market starts, this observation is intuitive. The price is informationally too efficient, in the sense that it already provides a sufficient statistic of the unknown process G. Therefore, investors do not use their private signals, which causes the classical no-trade situation in a rational expectations equilibrium.

This result is also consistent with Lemma 4 of Avdis and Glebkin (2025). In their model, investors actually trade, which is simply due to the fact that it is a one-period model. The optimal risky holdings are then allocated according to relative risk tolerance. The same effect is seen in our model. However, since our model is dynamic there is only one initial trade, which we may view as reflecting the trade situation in the one-period model. After the market has been established, investors no longer trade because they already hold the optimal share of risky assets.

As can be seen from the inequality in (5.1), price informativeness is minimized when the precision τ^2 is zero. Note that $\nu\colon [0,1) \to (0,\infty)$, so that $\tau^2 = 0$ is only possible if there is a non-trivial equilibrium satisfying $\mu = 0$. As mentioned in the beginning of this paper, we exclude this scenario because in this case the public signal only contains noise but no information about G. It is still worth mentioning two important points regarding the case $\tau^2 = 0$. First, under specific assumptions on the belief function β, a non-trivial equilibrium with $\mu = 0$ may indeed exist. Second, price informativeness is minimal because the public signal does not contain any information about G, but only accumulates individual signal noise.

Finally, let us consider the cases in between, i.e., whenever $0 < \tau^2 < \int_0^1 \nu(a)^2 \, \mathrm{d}a$. For this purpose, assume we have a non-trivial equilibrium as described in Proposition 3.5. Then, it holds that $\mu = \sigma_\xi^2 > 0$, and therefore $\tau^2 = \mu$. In particular, we have

$$\mu = \int_0^1 \nu(a) w(a) \, \mathrm{d}a = \int_{\{\beta \neq 0\}} \frac{\nu(a)^2}{1 + \tilde{\kappa} \frac{\psi(a, r v_0(a))}{\beta(a)}} \, \mathrm{d}a.$$

In general, we cannot make a statement about the extent to which a change in wealth, absolute risk aversion or beliefs affects the precision μ and therefore price informativeness $\mathcal{I}$. We have seen in Proposition 4.5 that scaling the belief function β also scales $\tilde{\kappa}$ to the same degree, while Corollary 4.6 shows that scaling absolute risk aversion ψ scales $\tilde{\kappa}$ inversely by the same amount. Consequently, scaling beliefs or absolute risk aversion does not change μ, and therefore price informativeness $\mathcal{I}$ remains unchanged. This makes intuitively sense because although beliefs or absolute risk aversion are adjusted, the relative structure between agents remains unchanged. At the same time, we have seen that the trading activity is proportional to the distance from a trivial equilibrium, which is denoted by $|\tilde{\kappa}|$. As a result, there is no clear relationship between trading activity and price informativeness either, as scaling beliefs or risk aversion affects trading activity but not price informativeness. We may thus conclude that although prices are the most informative in the presence of a no-trade equilibrium, there is no general relationship between trading activity and price informativeness.

5.2. Wealth effects

In more specific cases, we may still analyze how a given wealth allocation interacts with disagreement to determine price weights. For this purpose, we consider again a non-trivial equilibrium as described in Proposition 3.5. More precisely, we consider a weight function of the form

$$w(a) = \frac{\nu(a)}{1 + \tilde{\kappa}\frac{\psi(a, rv_0(a))}{\beta(a)}} \mathbb{1}_{\{\beta \neq 0\}}(a)$$

for some constant $\tilde{\kappa} \neq 0$. From the proof of Proposition 3.5, we know that $\tilde{\kappa} > 0$ whenever

$$\int_{\{\beta>0\}} \frac{\nu(a)^2 \psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a > -\int_{\{\beta<0\}} \frac{\nu(a)^2 \psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a. \tag{5.2}$$

Conversely, it holds that $\tilde{\kappa} < 0$ whenever the above strict inequality is reversed, so that

$$\int_{\{\beta>0\}} \frac{\nu(a)^2 \psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a < -\int_{\{\beta<0\}} \frac{\nu(a)^2 \psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a. \tag{5.3}$$

This fact has the following intuition. In the case where the strict inequality (5.2) holds true, there is more "positive" disagreement in the market (relative to the squared precision and absolute risk aversion). Since in a non-trivial equilibrium negative and positive disagreement have to be balanced out (see equation (3.6) in Proposition 3.4), the coefficient $\tilde{\kappa}$ has to be positive to reduce the weights of positively disagreeing agents. The same phenomenon occurs when there is more negative disagreement in the market. Then $\tilde{\kappa}$ has to be negative to reduce the weight of negatively disagreeing agents in order to achieve an equilibrium.

5.2.1. *The role of beliefs.* To analyze wealth effects and their interplay with disagreement in the market, we consider a *median wealth split* where the lower half is described by the subset $L \subseteq [0, 1)$, and the upper half is described by the subset $H \subseteq [0, 1)$. The chosen notation L and H shall refer to the terms "lower wealth" and "higher wealth", respectively. More precisely, it holds that $|L| = |H| = \frac{1}{2}$, and $\max_{a \in L} v_0(a) \leq \min_{a \in H} v_0(a)$. For any belief function $\beta \colon [0, 1) \to \mathbb{R}$, we further denote

$$H^+(\beta) \coloneqq H \cap \{\beta > 0\}, \qquad H^-(\beta) \coloneqq H \cap \{\beta < 0\},$$
$$L^+(\beta) \coloneqq L \cap \{\beta > 0\}, \qquad L^-(\beta) \coloneqq L \cap \{\beta < 0\}.$$

Hence, for a given belief function β the set $H^+(\beta)$ contains the wealthier agents with positive disagreement, while $H^-(\beta)$ contains the wealthier agents with negative disagreement. Let us further write the weight function as $w(a) = w_H(a) + w_L(a)$, where we define

$$w_H(a) := \frac{\nu(a)}{1 + \tilde{\kappa}\frac{\psi(a, rv_0(a))}{|\beta(a)|}} \mathbb{1}_{H^+(\beta)}(a) + \frac{\nu(a)}{1 - \tilde{\kappa}\frac{\psi(a, rv_0(a))}{|\beta(a)|}} \mathbb{1}_{H^-(\beta)}(a),$$
$$w_L(a) := \frac{\nu(a)}{1 + \tilde{\kappa}\frac{\psi(a, rv_0(a))}{|\beta(a)|}} \mathbb{1}_{L^+(\beta)}(a) + \frac{\nu(a)}{1 - \tilde{\kappa}\frac{\psi(a, rv_0(a))}{|\beta(a)|}} \mathbb{1}_{L^-(\beta)}(a).$$

In other words, the function $w_H \colon H \to \mathbb{R}$ describes the weights of the wealthier agents, while $w_L \colon L \to \mathbb{R}$ is the weight function of the less wealthy agents. To measure the reflection of private information of each group in the market clearing price we use the L^1-norms

$$\|w_H\|_1 := \int_0^1 |w_H(a)| \, \mathrm{d}a, \qquad \|w_L\|_1 := \int_0^1 |w_L(a)| \, \mathrm{d}a.$$

We then say that the private signals of the wealthier agents are more strongly reflected in the price whenever $\|w_H\|_1 > \|w_L\|_1$. Conversely, we say that the private signals of the less wealthy agents are more strongly reflected in the price whenever $\|w_H\|_1 < \|w_L\|_1$. Note that with this measure of signal reflection we can make statements on the aggregate market level but not on the individual investor level. However, we choose this measure because we are interested in an analysis of the aggregate market, and we do not want to compare individual agents. Moreover, this measure allows us to analyze more general situations, where investors in both group H and L may have positive and negative beliefs.

Proposition 5.1. *Assume that the functions $\beta(\cdot)$ and $\psi(\cdot, rv_0(\cdot))$ satisfy the assumptions of Proposition 3.5, and let $\tilde{\kappa} \neq 0$ be a solution of the corresponding fixed point problem. If the functions $\nu(\cdot)$, $\beta(\cdot)$, and $\psi(\cdot, rv_0(\cdot))$ are such that* (5.2) *implies*

$$\int_{\{\beta>0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a < - \int_{\{\beta<0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a,$$

and (5.3) *implies*

$$\int_{\{\beta>0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a > - \int_{\{\beta<0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a,$$

then the following assertions hold true:

(1) *If the less wealthy agents only have positive disagreement, while the wealthier agents only have negative disagreement, i.e., if $L^-(\beta) = H^+(\beta) = \emptyset$, and if in addition $\tilde{\kappa} < 0$ and $\min_{a \in L^+(\beta)} |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|} \geq \frac{1}{2}$, then $\|w_L\|_1 > \|w_H\|_1$.*

(2) If the less wealthy agents only have negative disagreement, while the wealthier agents only have positive disagreement, i.e., if $L^+(\beta) = H^-(\beta) = \emptyset$, and if in addition $\tilde{\kappa} < 0$ and $\min_{a \in H^+(\beta)} |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|} \geq \frac{1}{2}$, then $\|w_H\|_1 > \|w_L\|_1$.

An important takeaway from Proposition 5.1 is that private signals of wealthier agents are not necessarily more strongly reflected in the price, measured by the L^1-norm. A simple example is the case where the precision $\nu > 0$ is the same for all agents, and where we assume that $L^+(\beta) = \{\beta > 0\} = [0, \frac{1}{2})$ and $H^-(\beta) = \{\beta < 0\} = [\frac{1}{2}, 1)$ with $\beta(\cdot)$ and $\psi(\cdot, rv_0(\cdot))$ given by

$$\beta(a) = \beta_L \mathbb{1}_{[0,\frac{1}{2})}(a) - \beta_H \mathbb{1}_{[\frac{1}{2},1)}(a), \qquad \psi(a, rv_0(a)) = \psi_L \mathbb{1}_{[0,\frac{1}{2})}(a) + \psi_H \mathbb{1}_{[\frac{1}{2},1)}(a).$$

for some strictly positive constants β_L, β_H, ψ_L, and ψ_H. If also $\tilde{\kappa} < 0$, then (5.3) reads as

$$\frac{\nu^2 \psi_L}{2\beta_L} = \int_{\{\beta > 0\}} \frac{\nu(a)^2 \psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a < -\int_{\{\beta < 0\}} \frac{\nu(a)^2 \psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a = \frac{\nu^2 \psi_H}{2\beta_H},$$

which eventually implies

$$\int_{\{\beta > 0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a = \frac{\nu \beta_L}{2\psi_L} > \frac{\nu \beta_H}{2\psi_H} = -\int_{\{\beta < 0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a.$$

Hence, if also $\min_{a \in L^+(\beta)} |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|} \geq \frac{1}{2}$, then Proposition 5.1 implies $\|w_L\|_1 > \|w_H\|_1$. A similar example can also be found for the second assertion of Proposition 5.1.

Proposition 5.2. *Assume that the coefficients of absolute risk aversion are in a strict inverse monotonic relationship with the wealth distribution across agents, i.e., $v_0(a) > v_0(a')$ implies $\psi(a, rv_0(a)) < \psi(a', rv_0(a'))$ for all $a, a' \in [0, 1)$.*

(1) If precisions and absolute beliefs are in a monotonic relationship with the wealth distribution across agents, i.e., $v_0(a) \geq v_0(a')$ implies $\nu(a) \geq \nu(a')$ and $|\beta(a)| \geq |\beta(a')|$ for all $a, a' \in [0, 1)$, then the following assertions hold true:

(a) If the less wealthy agents only have negative disagreement, i.e., if $L^+(\beta) = \emptyset$, and if in addition $\tilde{\kappa} < 0$, then $\|w_H\|_1 > \|w_L\|_1$.

(b) If the less wealthy agents only have positive disagreement, i.e., if $L^-(\beta) = \emptyset$, and if in addition $\tilde{\kappa} > 0$, then $\|w_H\|_1 > \|w_L\|_1$.

(2) If absolute beliefs are in an inverse monotonic relationship with the wealth distribution across agents, i.e., $v_0(a) \geq v_0(a')$ implies $|\beta(a)| \leq |\beta(a')|$ for all $a, a' \in [0, 1)$, and if additionally

$$\min_{a \in L(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} > \max_{a \in H(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}},$$

then the following assertions hold true:

(a) If the wealthier agents only have negative disagreement, i.e., if $H^+(\beta) = \emptyset$, and if in addition $\tilde{\kappa} < 0$, then $\|w_L\|_1 > \|w_H\|_1$.

(b) If the wealthier agents only have positive disagreement, i.e., if $H^-(\beta) = \emptyset$, and if in addition $\tilde{\kappa} > 0$, then $\|w_L\|_1 > \|w_H\|_1$.

Remark 5.3. Note that the second property of Proposition 5.2 also holds true without any assumptions on the coefficients of absolute risk aversion.

The result of Proposition 5.2 is more specific. We are considering a situation in which absolute risk aversion decreases with increasing wealth across agents. The first assertion of Proposition 5.2 essentially states that under specific assumptions the private information of wealthier agents with stronger absolute beliefs and higher signal precisions is more strongly reflected in the price. In contrast, the second assertion of Proposition 5.2 states that, again under specific assumptions, the private information of the less wealthy agents with stronger absolute beliefs is more strongly reflected in the price. Hence, although these statements are not a universal statement, we observe that higher wealth alone is not enough to draw any conclusions about the strengths of the weights. In fact, the belief structure and also the precision of the private signals play a significant role. This fact stands in contrast to the one-period model of Avdis and Glebkin (2025), where the signals of wealthier agents are generally more strongly reflected in the price.

Example 5.4. We conclude this section with a specific example that shows the consequences of Proposition 5.2. We consider an economy with the following assumptions: (1) All agents have the same precision ν, (2) agents located on the interval $[0, \frac{1}{2})$ all have the same positive disagreement $\beta_1 > 0$, (3) agents located on the interval $[\frac{1}{2}, 1)$ all have the same negative disagreement $-\beta_2 < 0$, and (4) all agents have the same DARA utility function, so that the coefficient of absolute risk aversion satisfies $\frac{\partial \psi(a,x)}{\partial x} < 0$ for all $a \in [0, 1)$. Under these assumptions, it solely depends on the wealth distribution $v_0 \colon [0, 1) \to \mathbb{R}_+$ and the disagreement $\beta \colon [0, 1) \to \mathbb{R}$, whether we have more positive disagreement as in (5.2), or more negative disagreement as in (5.3). In fact, in this specific example, we have the following general rule to determine the sign of $\tilde{\kappa}$:

$$\tilde{\kappa} \begin{cases} < 0 & \text{if } \frac{\int_0^{\frac{1}{2}} \psi(rv_0(a))\,\mathrm{d}a}{\int_{\frac{1}{2}}^1 \psi(rv_0(a))\,\mathrm{d}a} < \frac{\beta_1}{\beta_2}, \\ > 0 & \text{if } \frac{\int_0^{\frac{1}{2}} \psi(rv_0(a))\,\mathrm{d}a}{\int_{\frac{1}{2}}^1 \psi(rv_0(a))\,\mathrm{d}a} > \frac{\beta_1}{\beta_2}. \end{cases} \tag{5.4}$$

In relation to the four assertions of Proposition 5.2, we consider the following four possible scenarios: (1a) wealthier investors are located on the interval $[0, \frac{1}{2})$ and $\beta_1 > \beta_2$, (1b) wealthier investors are located on the interval $[\frac{1}{2}, 1)$ and $\beta_1 < \beta_2$, (2a) wealthier investors are located on the interval $[\frac{1}{2}, 1)$ and $\beta_1 > \beta_2$, and (2b) wealthier investors are located on the interval $[0, \frac{1}{2})$ and $\beta_1 < \beta_2$.

In scenario (1a) we have $H(\beta) = [0, \frac{1}{2})$ and $L(\beta) = [\frac{1}{2}, 1)$. Since $\beta_1 > \beta_2$, and because all investors have the same DARA utility function, (5.4) yields $\tilde{\kappa} < 0$. Together with the fact that $L^+(\beta) = \emptyset$, it follows from Proposition 5.2 that $\|w_H\|_1 > \|w_L\|_1$. In scenario (1b) we have $H(\beta) = [\frac{1}{2}, 1)$ and $L(\beta) = [0, \frac{1}{2})$. Since $\beta_2 > \beta_1$, we now find from (5.4) that $\tilde{\kappa} > 0$, and since $L^-(\beta) = \emptyset$, Proposition 5.2 again implies $\|w_H\|_1 > \|w_L\|_1$.

Let us turn to scenarios (2a) and (2b). To avoid unnecessary complications in our analysis of these two scenarios, let us additionally assume that wealth is of the form

$$v_0(a) = v_0^l \mathbb{1}_{L(\beta)}(a) + v_0^h \mathbb{1}_{H(\beta)}(a), \qquad v_0^h > v_0^l. \tag{5.5}$$

In other words, wealth is constant in each income class. Now, in scenario (2a) we have $H(\beta) = [\frac{1}{2}, 1)$ and $L(\beta) = [0, \frac{1}{2})$, and we assume that $\beta_1 > \beta_2$. Together with equation (5.5), we find that according to (5.4) it holds that $\tilde{\kappa} < 0$, if and only if $\frac{\psi(rv_0^l)}{\beta_1} < \frac{\psi(rv_0^h)}{\beta_2}$. Consequently, if $\tilde{\kappa} < 0$, then it must hold

$$\min_{a \in L(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} = \frac{\nu}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0^l)}{\beta_1}} > \frac{\nu}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0^h)}{\beta_2}} = \max_{a \in H(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}},$$

and so Proposition 5.2 implies that $\|w_L\|_1 > \|w_H\|_1$. On the other hand, in scenario (2b) we have $H(\beta) = [0, \frac{1}{2})$ and $L(\beta) = [\frac{1}{2}, 1)$ and $\beta_1 < \beta_2$. Again by (5.4) and (5.5) we find that $\tilde{\kappa} > 0$ if and only if $\frac{\psi(rv_0^l)}{\beta_2} < \frac{\psi(rv_0^h)}{\beta_1}$. Hence, if $\tilde{\kappa} > 0$, then

$$\min_{a \in L(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} = \frac{\nu}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0^l)}{\beta_2}} > \frac{\nu}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0^h)}{\beta_1}} = \max_{a \in H(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}},$$

and thus by Proposition 5.2 it holds $\|w_L\|_1 > \|w_H\|_1$.

For the cases (2a) and (2b) it is worth noting that wealthier investors still can have stronger weights, which happens when the sign of $\tilde{\kappa}$ is flipped. Indeed, in case (2a) with $\tilde{\kappa} > 0$, the corresponding weight functions are of the form

$$w_L(a) = \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(rv_0^l)}{\beta_1}} \mathbb{1}_{[0, \frac{1}{2})}(a), \qquad w_H(a) = \frac{\nu(a)}{1 - |\tilde{\kappa}| \frac{\psi(rv_0^h)}{\beta_2}} \mathbb{1}_{[\frac{1}{2}, 1)}(a).$$

However, we find from (5.4) that $\tilde{\kappa} > 0$ if and only if $\frac{\psi(rv_0^l)}{\beta_1} > \frac{\psi(rv_0^h)}{\beta_2}$, which implies the reverse strict inequality $\|w_L\|_1 < \|w_H\|_1$. The argumentation for the case (2b) is done analogously.

5.2.2. *Reallocation of wealth and its effect on price informativeness.* We now turn to the question how a given wealth allocation relates to price informativeness, and how a reallocation of total wealth may change market efficiency. To understand the effect of a wealth reallocation on price informativeness, and how the belief function plays a role in this relationship, we have to consider the effect on total precision μ.

Proposition 5.5. *Assume that the functions $\beta(\cdot)$ and $\psi(\cdot, rv_0(\cdot))$ satisfy the assumptions of Proposition 3.5, and let $\tilde{\kappa}^{\text{old}} \neq 0$ be a solution of the corresponding fixed point problem. Consider a reallocation of wealth from $v_0^{\text{old}}\colon [0,1) \to \mathbb{R}_+$ to $v_0^{\text{new}}\colon [0,1) \to \mathbb{R}_+$, which satisfies $v_0^{\text{old}}(a) \geq v_0^{\text{new}}(a)$ for all $a \in H(\beta)$, and $v_0^{\text{old}}(a) \leq v_0^{\text{new}}(a)$ for all $a \in L(\beta)$. Assume that under the new wealth allocation a solution $\tilde{\kappa}^{\text{new}} \neq 0$ exists with $\operatorname{sgn}(\tilde{\kappa}^{\text{new}}) = \operatorname{sgn}(\tilde{\kappa}^{\text{old}})$. Further, assume that all investors have DARA utilities, i.e., $\frac{\partial \psi(a,x)}{\partial x} < 0$ for all $a \in [0,1)$. Then, the following assertions hold true:*

(1) Assume that $L^-(\beta) = H^+(\beta) = \emptyset$, and $\tilde{\kappa}^{\text{old}} < 0$:

(a) If $\tilde{\kappa}^{\text{old}} \leq \tilde{\kappa}^{\text{new}}$, $\max_{a\in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} < 1$, and

$$\max_{a\in H(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \leq \min_{a\in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|},$$

then $\mu^{\text{old}} \geq \mu^{\text{new}}$, and therefore $\mathcal{I}^{\text{old}} \geq \mathcal{I}^{\text{new}}$.

(b) If $\tilde{\kappa}^{\text{old}} \geq \tilde{\kappa}^{\text{new}}$, $\max_{a\in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} < 1$, and

$$\min_{a\in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \geq \max_{a\in L(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|},$$

then $\mu^{\text{old}} \geq \mu^{\text{new}}$, and therefore $\mathcal{I}^{\text{old}} \geq \mathcal{I}^{\text{new}}$.

(2) Assume $H^-(\beta) = L^+(\beta) = \emptyset$, and $\tilde{\kappa}^{\text{old}} < 0$:

(a) If $\tilde{\kappa}^{\text{old}} \leq \tilde{\kappa}^{\text{new}}$, $\max_{a\in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} < 1$, and

$$\max_{a\in H(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \leq \min_{a\in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|},$$

then $\mu^{\text{old}} \leq \mu^{\text{new}}$, and therefore $\mathcal{I}^{\text{old}} \leq \mathcal{I}^{\text{new}}$.

(b) If $\tilde{\kappa}^{\text{old}} \geq \tilde{\kappa}^{\text{new}}$, $\max_{a\in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} < 1$, and

$$\min_{a\in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \geq \max_{a\in L(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|},$$

then $\mu^{\text{old}} \leq \mu^{\text{new}}$, and therefore $\mathcal{I}^{\text{old}} \leq \mathcal{I}^{\text{new}}$.

In comparison with the results from the previous section, Proposition 5.5 is more specific. The main reason is that a change of wealth in general also implies a change in the equilibrium, which particularly results in a change of the parameter $\tilde{\kappa}$. However, even under more specific assumptions we observe that a wealth reallocation by transferring wealth from the wealthier agents to the less wealthy investors may both increase and decrease price informativeness. This result stands again in contrast to the one-period model of Avdis and Glebkin (2025), where transferring wealth from the wealthier to the less wealthy population increases informational efficiency in general.

6. Conclusion

This paper suggests a new conceptual approach to describe a fully dynamic rational expectations equilibrium with trade, and allows investors to have general risk-averse preferences without assuming a specific utility function. The underlying idea is to use a continuous economy with infinitely many agents that is obtained as a limiting discrete economy where investors agree to disagree about the distributional properties of the price. The key object is a Brownian sheet that allows us to describe information dynamics in a doubly continuous model, evolving in agent-time space. Aggregate demand is described by a two-parameter stochastic integral that behaves as a bridge-like stochastic process in agent direction, and therefore ensures market clearing with trade.

We show that a no-trade equilibrium always exists, while a trade equilibrium requires sufficiently dispersed disagreement. In particular, investors must have individual beliefs that allow for enough disagreement. In the presence of a trade equilibrium, we measure the distance to a no-trade equilibrium by the absolute value of the constant $\tilde{\kappa}$, which is part of the endogenously derived weight function via market clearing. We observe that the trading activity is in a monotonic relationship with the distance to a no-trade equilibrium: the larger the distance to a no-trade equilibrium, the higher the trading activity. Consequently, the trading activity is zero in a no-trade equilibrium.

The model also clarifies the role of wealth-dependent risk aversion. In equilibrium, wealth affects trading through absolute risk tolerance. Agents with higher risk tolerance hold larger average risky positions and react more strongly to stochastic information and disagreement. Under DARA, this implies that wealthier agents play a larger role in both risk sharing and trading activity. Under CARA, this channel is absent, which explains why the standard CARA benchmark cannot address this aspect of equilibrium trading.

The price informativeness is in an inversely monotonic relationship with the mean squared error of the public filter of the short term dividend mean, and therefore monotonic to the quality of information. More precisely, if the price reveals a better public filter, measured by the mean squared error, it concurrently contains more new information. Conversely, if the quality of public information decreases, price informativeness decreases as well.

Finally, these channels do not combine into general comparative statics. There is no systematic relationship between price informativeness, trading activity, wealth allocations, individual beliefs, and absolute risk aversions. In particular, the wealth channel above does not extend to the informational content of prices: even under very specific assumptions, we can show that wealthier agents' private information is not necessarily more strongly reflected in the price, because the belief structure plays a significant role here. The same is true for any wealth reallocation from the wealthier population to less wealthy agents: price informativeness may increase or decrease, depending on the belief structure.

Appendix

The appendix is organized in the following way. Appendix A summarizes important properties of the Brownian sheet. Appendix B provides a detailed explanation of the discretization of the continuous economy as described in Section 2. Appendix C covers all results that are related to the filtering equations, including the representation of the inference wedge $\Lambda^{a_i,n}$. In Appendix D we describe the measure change, which is needed to describe disagreement in our model. Appendix E covers all results related to the convergence of the solutions to the HJB equations of the discrete economies. In Appendix F we give all the proofs to the results of Section 3, while the proofs of Section 4 and Section 5 are given in Appendix G.

A. The Brownian sheet

This appendix collects the properties of the Brownian sheet that we use, stated so as to parallel the corresponding properties of Brownian motion. Throughout, for a rectangle $R = [a_1, a_2) \times [t_1, t_2) \subseteq [0, 1) \times \mathbb{R}_+$ we write

$$\Delta_R W := W_{(a_2,t_2)} - W_{(a_1,t_2)} - W_{(a_2,t_1)} + W_{(a_1,t_1)} \tag{A.1}$$

for the increment of W over R, and $|R| = (a_2 - a_1)(t_2 - t_1)$ for its area. Unlike the increment of a one-parameter process, $\Delta_R W$ is an alternating sum over the four corners of R. Figure 1 illustrates (A.1). Each term $W_{(a,t)}$ is the noise accumulated over the rectangle $[0, a) \times [0, t)$ anchored at the origin. Subtracting the two rectangles $[0, a_1) \times [0, t_2)$ and $[0, a_2) \times [0, t_1)$ from $[0, a_2) \times [0, t_2)$ removes their common part $[0, a_1) \times [0, t_1)$ twice, so it is added back once.

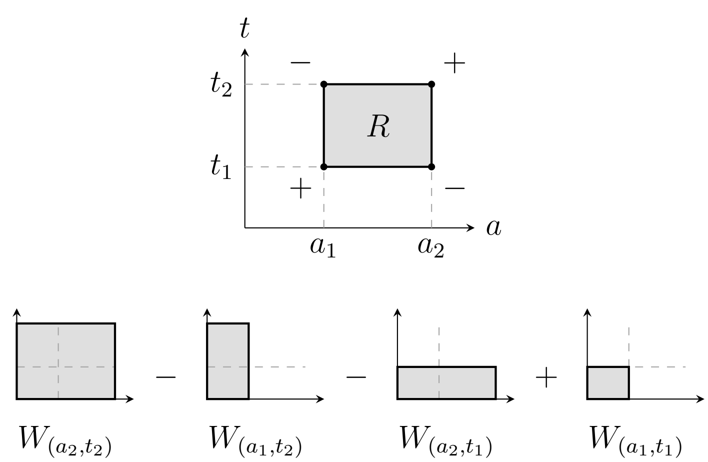

Figure 1: The increment $\Delta_R W$ of a Brownian sheet over the rectangle $R = [a_1, a_2) \times [t_1, t_2)$. Top: R and the signs with which the four corners enter (A.1). Bottom: the same identity read as inclusion–exclusion among rectangles anchored at the origin.

A *Brownian sheet* $W = (W_{(a,t)})_{(a,t)\in[0,1)\times\mathbb{R}_+}$ has the following characteristic properties:

(i) $W_{(a,0)} = W_{(0,t)} = 0$ for all a and t;

(ii) the increments $\Delta_{R_1}W, \dots, \Delta_{R_n}W$ are independent whenever the rectangles $R_1, \dots, R_n$ have disjoint interiors;

(iii) $\Delta_R W \sim \mathcal{N}(0, |R|)$ for every rectangle R;

(iv) W has continuous sample paths.

These properties are the two-parameter analogue of the standard definition of Brownian motion, with intervals replaced by rectangles and length replaced by area:

		Brownian motion	**Brownian sheet**
(i)	initialization	$W_0 = 0$	$W_{(a,0)} = W_{(0,t)} = 0$
(ii)	independence	over disjoint intervals	over non-overlapping rectangles
(iii)	increments	$\mathcal{N}(0, \text{length})$	$\mathcal{N}(0, \text{area})$
(iv)	regularity	continuous paths	continuous paths

We comment on a slight difference in (i). For Brownian motion the increments determine the process up to an additive constant, which is why it suffices to fix its value at a single point. For a Brownian sheet the rectangular increments determine the field only up to an additive term of the form $f(a) + g(t)$: a direct application of (A.1) yields

$$\Delta_R\big(W + f(a) + g(t)\big) = \Delta_R W \qquad \text{for every rectangle } R,$$

because each of the values $f(a_1)$, $f(a_2)$, $g(t_1)$ and $g(t_2)$ enters the alternating sum once with each sign. Pinning the field down therefore requires fixing it on both axes rather than at the origin alone.

Fixing one parameter reduces the sheet to a one-parameter process, which is again a Brownian motion after rescaling.

Corollary A.1. *Let W be a Brownian sheet. For every fixed $a > 0$ the process $(W_{(a,t)})_{t\geq 0}$ is a Brownian motion scaled by $\sqrt{a}$; that is, $B^a := (a^{-1/2}W_{(a,t)})_{t\geq 0}$ is a standard Brownian motion. Symmetrically, for every fixed $t > 0$ the process $(t^{-1/2}W_{(a,t)})_{a\in[0,1)}$ is a standard Brownian motion.*

Proof. Fix $a > 0$ and consider $R = [0, a) \times [t_1, t_2)$ for $0 \leq t_1 < t_2$. By part (i) the two corners on the axis $\{a = 0\}$ vanish, so (A.1) collapses to a two-point difference,

$$\Delta_R W = W_{(a,t_2)} - W_{(0,t_2)} - W_{(a,t_1)} + W_{(0,t_1)} = W_{(a,t_2)} - W_{(a,t_1)}.$$

Hence $W_{(a,t_2)} - W_{(a,t_1)} = \Delta_R W \sim \mathcal{N}(0, a(t_2 - t_1))$ by part (iii), so $B^a_{t_2} - B^a_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$. Disjoint time intervals $[t_1, t_2)$ give rectangles $[0, a) \times [t_1, t_2)$ with disjoint interiors, so these increments are independent by part (ii). Finally $B^a_0 = a^{-1/2} W_{(a,0)} = 0$ by part (i), and B^a has continuous paths by part (iv). Thus B^a is a standard Brownian motion. The statement for fixed t follows by exchanging the roles of the two coordinates. ∎

Aggregating a sheet across agents against a square-integrable weight likewise produces an ordinary one-parameter Brownian motion. We first record the isometry on which the construction of the two-parameter stochastic integral rests.

Lemma A.2 (Itô isometry)**.** *Let W be a Brownian sheet and let $f \in L^2([0,1) \times [0,T))$. Then $\int_{[(0,0),(1,T))} f(a,s)\,\mathrm{d}W_{(a,s)}$ is well defined, has mean zero, and*

$$\mathrm{E}\left[\left(\int_{[(0,0),(1,T))} f(a,s)\,\mathrm{d}W_{(a,s)}\right)^2\right] = \int_0^1 \int_0^T f(a,s)^2\,\mathrm{d}s\,\mathrm{d}a.$$

We refer to Walsh (1986) for the construction of this integral.

Proposition A.3. *Let W be a Brownian sheet and let $w \in L^2([0,1))$ with $\sigma_w^2 := \int_0^1 w(a)^2\,\mathrm{d}a > 0$. Then*

$$\int_{[(0,0),(1,t))} w(a)\,\mathrm{d}W_{(a,s)}, \qquad t \geq 0,$$

is a Brownian motion scaled by σ_w; that is, $\sigma_w^{-1} \int_{[(0,0),(1,\cdot))} w(a)\,\mathrm{d}W_{(a,s)}$ is a standard Brownian motion. In particular $\int_{[(0,0),(1,t))} w(a)\,\mathrm{d}W_{(a,s)} \sim \mathcal{N}(0, t\sigma_w^2)$.

Proof. Let first $w = \sum_{i=0}^{n-1} w_i \mathbb{1}_{[a_i, a_{i+1})}$ be a step function with $0 = a_0 < \cdots < a_n = 1$. For $0 \leq t_1 < t_2$ set $R_i := [a_i, a_{i+1}) \times [t_1, t_2)$, so that the increment over the time interval $[t_1, t_2)$ is

$$\int_{[(0,t_1),(1,t_2))} w(a)\,\mathrm{d}W_{(a,s)} = \sum_{i=0}^{n-1} w_i \Delta_{R_i} W.$$

The rectangles $R_0, \ldots, R_{n-1}$ have disjoint interiors, so by parts (ii) and (iii) the summands are independent with $\Delta_{R_i} W \sim \mathcal{N}\big(0, (a_{i+1} - a_i)(t_2 - t_1)\big)$, and therefore

$$\int_{[(0,t_1),(1,t_2))} w(a)\,\mathrm{d}W_{(a,s)} \sim \mathcal{N}\Big(0, (t_2 - t_1) \textstyle\sum_i w_i^2 (a_{i+1} - a_i)\Big) = \mathcal{N}\big(0, (t_2 - t_1)\sigma_w^2\big).$$

Increments over disjoint time intervals are independent, again by part (ii), since the associated rectangles have disjoint interiors. Together with $\int_{[(0,0),(1,0))} w(a)\,\mathrm{d}W_{(a,s)} = 0$ and the continuity of W, this proves the claim for step functions. For general $w \in L^2([0,1))$ choose step functions $w^n \to w$ in $L^2([0,1))$; by Lemma A.2 the associated processes converge in L^2, and the limit inherits the stated properties. ∎

B. Discretization of the continuous economy

This appendix contains the detailed derivation to obtain the desired discretized economy as discussed in Section 2.

B.1. Discretized information dynamics

To discretize private information, we note that a discrete increment of the process ξ over the rectangle $[(a_i, t_j), (a_{i+1}, t_{j+1}))$, with $t_j < t_{j+1}$, is of the form

$$\Delta\xi_{(a_i,t_j)} = \nu(a_i)\Delta G_{t_j}\Delta a_i + \Delta W^{\xi}_{(a_i,t_j)},$$

where $\Delta G_{t_j} = G_{t_{j+1}} - G_{t_j}$ and $\Delta W^{\xi}_{(a_i,t_j)} = W^{\xi}_{(a_{i+1},t_{j+1})} - W^{\xi}_{(a_{i+1},t_j)} - W^{\xi}_{(a_i,t_{j+1})} + W^{\xi}_{(a_i,t_j)}$. By assuming that $(t_j)_{0\le j\le m}$ forms a partition of the interval $[0,t]$, summing over $0 \le j \le m$ yields

$$\Delta_1\xi_{(a_i,t)} - \Delta_1\xi_{(a_i,0)} = \nu(a_i)(G_t - G_0)\Delta a_i + \Delta_1 W^{\xi}_{(a_i,t)},$$

where we denote $\Delta_1 W^{\xi}_{(a_i,t)} = W^{\xi}_{(a_{i+1},t)} - W^{\xi}_{(a_i,t)}$. To ease notation, and to distinguish between the discrete economy and the continuous economy, we denote the discretized private signals by $\zeta^{a_i,n}_t := \Delta_1\xi_{(a_i,t)}$. In particular, $\zeta^{a_i,n} = (\zeta^{a_i,n}_t)_{t\ge 0}$ describes the private signal of investor a_i in the discrete economy with n investors. We further define the process $W^{\zeta^{a_i,n}} = (W^{\zeta^{a_i,n}}_t)_{t\ge 0}$ via

$$W^{\zeta^{a_i,n}}_t := (\Delta a_i)^{-1/2}\Delta_1 W^{\xi}_{(a_i,t)} = m^{-1}_{a_i,n}\Delta_1 W^{\xi}_{(a_i,t)}, \qquad m_{a_i,n} := (\Delta a_i)^{1/2},$$

which is a Brownian motion in t-direction. The process $\zeta^{a_i,n}$ then satisfies the stochastic differential equation

$$\mathrm{d}\zeta^{a_i,n}_t = \overline{\nu}_{a_i,n}\,\mathrm{d}G_t + m_{a_i,n}\,\mathrm{d}W^{\zeta^{a_i,n}}_t, \qquad \overline{\nu}_{a_i,n} := \nu(a_i)\Delta a_i.$$

In particular, we may consider the discretized precision function $\nu^n\colon [0,1) \to (0,\infty)$, which is defined as the simple function

$$\nu^n(a) := \sum_{i=0}^{n-1} \nu(a_i)\mathbb{1}_{[a_i,a_{i+1})}(a),$$

and then define $\xi^n = (\xi^n_{(a,t)})_{(a,t)\in[0,1)\times\mathbb{R}_+}$ via

$$\mathrm{d}\xi^n_{(a,t)} = \nu^n(a)\,\mathrm{d}G_t\,\mathrm{d}a + \mathrm{d}W^{\xi}_{(a,t)}.$$

The discretized private signal then satisfies

$$\zeta_t^{a_i,n} = \int_{[(a_i,0),(a_{i+1},t))} \mathrm{d}\xi^n_{(a,t)} = \left(\int_{a_i}^{a_{i+1}} \nu^n(a)\,\mathrm{d}a\right)(G_t - G_0) + \int_{[(a_i,0),(a_{i+1},t))} \mathrm{d}W^\xi_{(a,t)}.$$

The discretization of the precision function makes sense because investor a_i "lives" on the interval $[a_i, a_{i+1})$, and thus should have the same precision on this entire interval.

B.2. Discretized public information

To obtain a discrete version of the public signal we also discretize the weight function by considering the simple function

$$w^n(a) \coloneqq \sum_{i=0}^{n-1} w(a_i)\mathbb{1}_{[a_i,a_{i+1})}(a).$$

Public information is then described by the two-parameter stochastic integral of the private-signal field ξ^n against the weight function w^n, which we will denote by Ξ^n, and which satisfies

$$\Xi^n_t \coloneqq \int_{[(0,0),(1,t))} w^n(a)\,\mathrm{d}\xi^n_{(a,s)} = \mu_n(G_t - G_0) + \int_{[(0,0),(1,t))} w^n(a)\,\mathrm{d}W^\xi_{(a,s)},$$

where we denote $\mu_n \coloneqq \int_0^1 \nu^n(a)w^n(a)\,\mathrm{d}a$. To write the public signal as a linear Itô process, let us define the process $W^{\Xi,n} = (W^{\Xi,n}_t)_{t\geq 0}$ via

$$W^{\Xi,n}_t \coloneqq \sigma^{-1}_{\xi,n} \int_{[(0,0),(1,t))} w^n(a)\,\mathrm{d}W^\xi_{(a,s)}, \qquad \sigma_{\xi,n} \coloneqq \left(\int_0^1 w^n(a)^2\,\mathrm{d}a\right)^{1/2},$$

which is a Brownian motion in t-direction. The public signal $\Xi^n = (\Xi^n_t)_{t\geq 0}$ then satisfies the stochastic differential equation

$$\mathrm{d}\Xi^n_t = \mu_n\,\mathrm{d}G_t + \sigma_{\xi,n}\,\mathrm{d}W^{\Xi,n}_t.$$

B.3. Agree to disagree

Investors agree to disagree about the price dynamics in the discrete economy. To understand the nature of the disagreement, one may observe that the dynamics of the risky asset in the n-th discrete economy satisfies the stochastic differential equation

$$\mathrm{d}S^n_t = \begin{pmatrix} s_G\alpha_G g \\ -s_D\alpha_D \\ s_D\alpha_D - s_G\alpha_G \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 1 \\ D_t \\ \widehat{G}^n_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} s_D + s_G\Gamma^n_1 \\ s_G\Gamma^n_2 \end{pmatrix}^{\mathsf{T}} \boldsymbol{Q}^n\,\mathrm{d}\underline{\widehat{W}}^n_t, \tag{B.1}$$

where $\boldsymbol{Q}^n \in \mathbb{R}^{2\times 2}$ is symmetric and non-singular, $\underline{\widehat{W}}^n = (\underline{\widehat{W}}_t^n)_{t\geq 0}$ is a $\mathbb{G}^n$-Brownian motion, and Γ_1^n and Γ_2^n are constants obtained from filtering. More precisely, we have

$$\boldsymbol{Q}^n := \begin{pmatrix} \sigma_D & 0 \\ 0 & (\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2)^{1/2} \end{pmatrix}, \quad \Gamma_1^n := \frac{\alpha_D\gamma^n}{\sigma_D^2}, \quad \Gamma_2^n := \frac{\mu_n(\sigma_G^2 - \alpha_G\gamma^n)}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2},$$

where $\gamma^n = \lim_{t\to\infty}\gamma_t^n$ is the steady state solution of the mean squared filtering error, i.e., $\gamma_t^n = \mathrm{E}[(\widehat{G}_t^n - G_t)^2]$ (see Lemma C.2 together with Lemma C.11 for details). In fact, each investor may rewrite this stochastic differential equation to obtain the price dynamics as a linear Itô process with a Brownian motion that is adapted to the private filtration $\mathbb{G}^{a_i,n} = (\mathcal{G}_t^{a_i,n})_{t\geq 0}$, where $\mathcal{G}_t^{a_i,n} = \sigma(D_s, \Xi_s^n, \zeta_s^{a_i,n};\ s \leq t)$. In particular, let

$$\widehat{G}_t^{a_i,n} := \mathrm{E}[G_t|\mathcal{G}_t^{a_i,n}] \qquad \text{and} \qquad \Lambda_t^{a_i,n} := \widehat{G}_t^{a_i,n} - \widehat{G}_t^n$$

be the *inference* and *inference wedge* (wedge, for short) of agent a_i, respectively, in the economy with n agents at time t. We can then rewrite the price dynamics (B.1) as

$$\mathrm{d}S_t^n = \begin{pmatrix} s_G\alpha_G g \\ -s_D\alpha_D \\ s_D\alpha_D - s_G\alpha_G \\ s_G\alpha_\Lambda^n \end{pmatrix}^\intercal \begin{pmatrix} 1 \\ D_t \\ \widehat{G}_t^{a_i,n} \\ \Lambda_t^{a_i,n} \end{pmatrix} \mathrm{d}t + \begin{pmatrix} s_D + s_G\Gamma_1^n \\ s_G\Gamma_2^n \\ 0 \end{pmatrix}^\intercal \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_t^{a_i,n}, \tag{B.2}$$

where $\alpha_\Lambda^n := \alpha_D\Gamma_1^n + \alpha_G(1-\mu_n\Gamma_2^n)$, where $\boldsymbol{Q}^{a_i,n} \in \mathbb{R}^{3\times 3}$ is symmetric and non-singular, and where $\underline{\widehat{W}}^{a_i,n} = (\underline{\widehat{W}}_t^{a_i,n})_{t\geq 0}$ is a $\mathbb{G}^{a_i,n}$-Brownian motion (see Lemma C.1 and Corollary C.3 together with Lemma C.11).

By comparing the stochastic differential equations (B.1) and (B.2), we observe that passing to the private filtration $\mathbb{G}^{a_i,n}$ adds a new drift term $s_G\alpha_\Lambda^n\Lambda_t^{a_i,n}$, which eventually converges to 0 as $n\to\infty$. We now assume that investors agree to disagree about the extent to which their difference of filtering $\Lambda^{a_i,n}$ enters the price in the discrete economy. We achieve this disagreement by replacing the term $s_G\alpha_\Lambda^n$ by an investor-specific constant $\tilde{\beta}(a_i)$, where $\tilde{\beta}\colon [0,1) \to \mathbb{R}$ satisfies suitable regularity assumptions. Mathematically speaking, the disagreement is achieved by an equivalent measure change via Girsanov transformation (see Appendix D for details). More precisely, each investor works under an equivalent probability measure $P^{a_i,n} \approx P$ under which the price process S^n satisfies the stochastic differential equation

$$\mathrm{d}S_t^n = \begin{pmatrix} s_G\alpha_G g \\ -s_D\alpha_D \\ s_D\alpha_D - s_G\alpha_G \\ \tilde{\beta}(a_i) \end{pmatrix}^\intercal \begin{pmatrix} 1 \\ D_t \\ \widehat{G}_t^{a_i,n} \\ \Lambda_t^{a_i,n} \end{pmatrix} \mathrm{d}t + \begin{pmatrix} s_D + s_G\Gamma_1^n \\ s_G\Gamma_2^n \\ 0 \end{pmatrix}^\intercal \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widetilde{W}}_t^{a_i,n}, \tag{B.3}$$

where $\widetilde{\underline{W}}^{a_i,n} = (\widetilde{\underline{W}}^{a_i,n}_t)_{t\geq 0}$ is a $P^{a_i,n}$-Brownian motion. It is worth noting that the density process $\frac{\mathrm{d}P^{a_i,n}}{\mathrm{d}P}$ converges to 1 as $n \to \infty$. Hence, there is no disagreement in the continuous economy.

B.4. Excess return

We conjecture that the state variables are given by the wealth $V^{a_i,n} = (V^{a_i,n}_t)_{t\geq 0}$ and the inference wedge $\Lambda^{a_i,n} = (\Lambda^{a_i,n}_t)_{t\geq 0}$. Equivalently, we conjecture that prices, dividends, and inferences do not enter the excess return. To see this fact, note that the instantaneous excess return in the n-th discrete economy satisfies $\mathrm{d}R^n_t = (D_t - rS^n_t)\,\mathrm{d}t + \mathrm{d}S^n_t$. Hence, by using the price conjecture (2.10) together with the price dynamics (B.3) we thus obtain

$$\mathrm{d}R^n_t = \begin{pmatrix} s_G\alpha_G g - s_0 r - r\frac{\rho}{n} \\ 1 - s_D(r+\alpha_D) \\ s_D\alpha_D - s_G(r+\alpha_G) \\ s_G r + \tilde{\beta}(a_i) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 1 \\ D_t \\ \widehat{G}^{a_i,n}_t \\ \Lambda^{a_i,n}_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} s_D + s_G\Gamma^n_1 \\ s_G\Gamma^n_2 \\ 0 \end{pmatrix}^{\mathsf{T}} \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\widetilde{\underline{W}}^{a_i,n}_t.$$

Hence, by setting

$$s_0 = \frac{s_G\alpha_G g}{r}, \qquad s_D = \frac{1}{r+\alpha_D}, \qquad s_G = \frac{s_D\alpha_D}{r+\alpha_G},$$

the instantaneous excess return in the n-th discrete economy simplifies to

$$\mathrm{d}R^n_t = \begin{pmatrix} -r\frac{\rho}{n} \\ s_G r + \tilde{\beta}(a_i) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 1 \\ \Lambda^{a_i,n}_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} s_D + s_G\Gamma^n_1 \\ s_G\Gamma^n_2 \\ 0 \end{pmatrix}^{\mathsf{T}} \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\widetilde{\underline{W}}^{a_i,n}_t,$$

and thus verifies our conjecture.

In terms of equation (2.12) in the main text, note that $\tilde{\beta}(a_i)$ replaces the initial drift $s_G\alpha^n_\Lambda$ from (B.2), when passing to the price dynamics (B.3) via Girsanov. Hence, if no measure change occurs, then $\tilde{\beta}(a_i)$ equals $s_G\alpha^n_\Lambda$, which explains the origin of $\beta^*_n = s_G(r+\alpha^n_\Lambda)$. In other words, the final object of interest is the instantaneous excess return R^n, and within R^n we are especially interested in the factor that loads on the inference wedge $\Lambda^{a_i,n}$. This factor has two sources. The term $s_G r$ comes from the drift part of the excess return, and is unaffected by the measure change which appears on the level of prices. On the other hand, the term $s_G\alpha^n_\Lambda$ comes from the price dynamics, and therefore is affected by the measure change. No measure change contributes $s_G\alpha^n_\Lambda$, yielding β^*_n, while a measure change yields $\tilde{\beta}(a_i)$, which instead of β^*_n then yields a factor $\beta(a_i) = s_G r + \tilde{\beta}(a_i)$.

C. Filtering

This appendix is concerned with all results that are related to filtering. The outline is as follows: Section C.1 covers the private filter $\widehat{G}^{a_i,n}$, the public filter $\widehat{G}^n$, and the dynamics of the inference wedge $\Lambda^{a_i,n}$. Section C.2 analyzes the coefficients occurring in the filtering equations obtained in Section C.1 and discusses their relation. Section C.3 explains the convergence of the filters obtained in the discrete economies to the filter of the continuous economy. Section C.4 provides a convergence result of the aggregate filter increments in the form of the inference wedge, which essentially is the main result of this appendix. Finally, Section C.5 provides auxiliary results related to filtering, including the representation as an Ornstein-Uhlenbeck process.

C.1. Filtering equations in the discrete economy

In this section we are mainly interested in the discrete-economy public filter $\widehat{G}^n$, the discrete-economy private filter $\widehat{G}^{a_i,n}$, and their difference $\Lambda^{a_i,n} = \widehat{G}^{a_i,n} - \widehat{G}^n$. The first result describes the private filter $\widehat{G}^{a_i,n}$.

Lemma C.1. *The process $\widehat{G}^{a_i,n} = (\widehat{G}^{a_i,n}_t)_{t\geq 0}$ defined via $\widehat{G}^{a_i,n}_t := \mathrm{E}[G_t|\mathcal{G}^{a_i,n}_t]$, with the filtration $\mathbb{G}^{a_i,n} = (\mathcal{G}^{a_i,n}_t)_{t\geq 0}$ given by $\mathcal{G}^{a_i,n}_t := \sigma(D_s, \Xi^n_s, \zeta^{a_i,n}_s;\, s \leq t)$, satisfies the stochastic differential equation*

$$\mathrm{d}\widehat{G}^{a_i,n}_t = \alpha_G(g - \widehat{G}^{a_i,n}_t)\,\mathrm{d}t + \begin{pmatrix} \Gamma^{a_i,n}_1(t) \\ \Gamma^{a_i,n}_2(t) \\ \Gamma^{a_i,n}_3(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}^{a_i,n}_t)\,\mathrm{d}t \\ \mathrm{d}\Xi^n_t - \mu_n\alpha_G(g - \widehat{G}^{a_i,n}_t)\,\mathrm{d}t \\ \mathrm{d}\zeta^{a_i,n}_t - \overline{\nu}_{a_i,n}\alpha_G(g - \widehat{G}^{a_i,n}_t)\,\mathrm{d}t \end{pmatrix}, \tag{C.1}$$

where the vector $\mathbf{\Gamma}^{a_i,n}_t = \begin{pmatrix} \Gamma^{a_i,n}_1(t) & \Gamma^{a_i,n}_2(t) & \Gamma^{a_i,n}_3(t) \end{pmatrix} \in \mathbb{R}^{1\times 3}$ is defined via

$$\mathbf{\Gamma}^{a_i,n}_t = \begin{pmatrix} \frac{\alpha_D\gamma^{a_i,n}_t}{\sigma^2_D} & \frac{(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\sigma^2_G - \alpha_G\gamma^{a_i,n}_t)}{\tilde{\mu}^2_{a_i,n}\sigma^2_G + \tilde{\sigma}^2_{a_i,n}} & \frac{(\nu(a_i)\sigma^2_{\xi,n} - w(a_i)\mu_n)(\sigma^2_G - \alpha_G\gamma^{a_i,n}_t)}{\tilde{\mu}^2_{a_i,n}\sigma^2_G + \tilde{\sigma}^2_{a_i,n}} \end{pmatrix}, \tag{C.2}$$

with the coefficients

$$\mu_n := \int_0^1 \nu^n(a)w^n(a)\,\mathrm{d}a, \qquad \sigma_{\xi,n} := \left(\int_0^1 w^n(a)^2\,\mathrm{d}a\right)^{1/2},$$

$$\tilde{\mu}^2_{a_i,n} := \mu^2_n - 2\mu_n w(a_i)\overline{\nu}_{a_i,n} + \sigma^2_{\xi,n}\frac{\overline{\nu}^2_{a_i,n}}{\Delta a_i}, \qquad \tilde{\sigma}^2_{a_i,n} := \sigma^2_{\xi,n} - w(a_i)^2\Delta a_i.$$

The mean squared error $\gamma^{a_i,n}_t := \mathrm{E}[(G_t - \widehat{G}^{a_i,n}_t)^2]$ satisfies the Riccati equation

$$\frac{\mathrm{d}\gamma^{a_i,n}_t}{\mathrm{d}t} = \frac{\tilde{\sigma}^2_{a_i,n}}{\tilde{\mu}^2_{a_i,n}\sigma^2_G + \tilde{\sigma}^2_{a_i,n}}(\sigma^2_G - 2\alpha_G\gamma^{a_i,n}_t) - \left(\frac{\alpha^2_D}{\sigma^2_D} + \frac{\alpha^2_G\tilde{\mu}^2_{a_i,n}}{\tilde{\mu}^2_{a_i,n}\sigma^2_G + \tilde{\sigma}^2_{a_i,n}}\right)(\gamma^{a_i,n}_t)^2. \tag{C.3}$$

Proof. Consider the observable process

$$\mathrm{d}\underline{Y}_t^{a_i,n} := \begin{pmatrix} \mathrm{d}D_t + \alpha_D D_t\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G g\,\mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G g\,\mathrm{d}t \end{pmatrix},$$

which satisfies the stochastic differential equation

$$\mathrm{d}\underline{Y}_t^{a_i,n} = \begin{pmatrix} \alpha_D \\ -\mu_n\alpha_G \\ -\overline{\nu}_{a_i,n}\alpha_G \end{pmatrix} G_t\,\mathrm{d}t + \begin{pmatrix} 0 \\ \mu_n\sigma_G \\ \overline{\nu}_{a_i,n}\sigma_G \end{pmatrix} \mathrm{d}W_t^G + \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & \sigma_{\xi,n} & 0 \\ 0 & 0 & m_{a_i,n} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^D \\ \mathrm{d}W_t^{\Xi,n} \\ \mathrm{d}W_t^{\zeta^{a_i,n}} \end{pmatrix}.$$

The process

$$\underline{W}_t^{Y^{a_i,n}} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{\sigma}_{a_i,n}^{-1} & 0 \\ 0 & 0 & m_{a_i,n}^{-1} \end{pmatrix} \begin{pmatrix} W_t^D \\ \int_{[(0,0),(1,t))} w^n(a)\,\mathrm{d}W_{(a,s)}^{\xi} - w(a_i)\Delta_1 W_{(a_i,t)}^{\xi} \\ \Delta_1 W_{(a_i,t)}^{\xi} \end{pmatrix}$$

is an $\mathbb{R}^3$-valued Brownian motion, and we may write

$$\begin{pmatrix} \mathrm{d}W_t^D \\ \mathrm{d}W_t^{\Xi,n} \\ \mathrm{d}W_t^{\zeta^{a_i,n}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{\sigma}_{a_i,n}\sigma_{\xi,n}^{-1} & \sigma_{\xi,n}^{-1} w(a_i) m_{a_i,n} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^D \\ \tilde{\sigma}_{a_i,n}^{-1}(\sigma_{\xi,n}\,\mathrm{d}W_t^{\Xi,n} - w(a_i) m_{a_i,n}\,\mathrm{d}W_t^{\zeta^{a_i,n}}) \\ \mathrm{d}W_t^{\zeta^{a_i,n}} \end{pmatrix},$$

so that

$$\begin{aligned} \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & \sigma_{\xi,n} & 0 \\ 0 & 0 & m_{a_i,n} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^D \\ \mathrm{d}W_t^{\Xi,n} \\ \mathrm{d}W_t^{\zeta^{a_i,n}} \end{pmatrix} &= \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & \sigma_{\xi,n} & 0 \\ 0 & 0 & m_{a_i,n} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\tilde{\sigma}_{a_i,n}}{\sigma_{\xi,n}} & \frac{w(a_i) m_{a_i,n}}{\sigma_{\xi,n}} \\ 0 & 0 & 1 \end{pmatrix} \mathrm{d}\underline{W}_t^{Y^{a_i,n}} \\ &= \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & \tilde{\sigma}_{a_i,n} & w(a_i) m_{a_i,n} \\ 0 & 0 & m_{a_i,n} \end{pmatrix} \mathrm{d}\underline{W}_t^{Y^{a_i,n}} =: \boldsymbol{\Sigma}_{Y^{a_i,n}}\,\mathrm{d}\underline{W}_t^{Y^{a_i,n}}. \end{aligned}$$

By denoting

$$\boldsymbol{A}_{Y^{a_i,n}} := \begin{pmatrix} \alpha_D \\ -\mu_n\alpha_G \\ -\overline{\nu}_{a_i,n}\alpha_G \end{pmatrix}, \qquad \boldsymbol{\Sigma}_G := \begin{pmatrix} 0 \\ \mu_n\sigma_G \\ \overline{\nu}_{a_i,n}\sigma_G \end{pmatrix}$$

we thus obtain $\mathrm{d}\underline{Y}_t^{a_i,n} = \boldsymbol{A}_{Y^{a_i,n}} G_t\,\mathrm{d}t + \boldsymbol{\Sigma}_G\,\mathrm{d}W_t^G + \boldsymbol{\Sigma}_{Y^{a_i,n}}\,\mathrm{d}\underline{W}_t^{a_i,n}$. Theorem 10.3 of Liptser and Shiryaev (2001) then yields

$$\mathrm{d}\widehat{G}_t^{a_i,n} = \alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t + \boldsymbol{\Gamma}_t^{a_i,n}[\mathrm{d}\underline{Y}_t^{a_i,n} - \boldsymbol{A}_{Y^{a_i,n}}\widehat{G}_t^{a_i,n}\,\mathrm{d}t],$$

and

$$\frac{\mathrm{d}\gamma_t^{a_i,n}}{\mathrm{d}t} = \sigma_G^2 - 2\alpha_G\gamma_t^{a_i,n} - \boldsymbol{\Gamma}_t^{a_i,n}\left[\begin{pmatrix}\sigma_G & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}} + \gamma_t^{a_i,n}\boldsymbol{A}_{Y^{a_i,n}}^{\mathsf{T}}\right]^{\mathsf{T}},$$

where

$$\boldsymbol{\Gamma}_t^{a_i,n} = \left[\begin{pmatrix}\sigma_G & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}} + \gamma_t^{a_i,n}\boldsymbol{A}_{Y^{a_i,n}}^{\mathsf{T}}\right]\left(\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}}\right)^{-1}.$$

To prove (C.1) it thus remains to compute $\boldsymbol{\Gamma}_t^{a_i,n}$. For this purpose we first note that

$$\begin{pmatrix}\sigma_G & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}} = \begin{pmatrix}\sigma_G\\ 0\\ 0\\ 0\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}0 & \mu_n\sigma_G & \overline{\nu}_{a_i,n}\sigma_G\\ \sigma_D & 0 & 0\\ 0 & \tilde{\sigma}_{a_i,n} & 0\\ 0 & w(a_i)m_{a_i,n} & m_{a_i,n}\end{pmatrix} = \begin{pmatrix}0\\ \mu_n\sigma_G^2\\ \overline{\nu}_{a_i,n}\sigma_G^2\end{pmatrix}^{\mathsf{T}},$$

which implies

$$\begin{pmatrix}\sigma_G\\ 0\\ 0\\ 0\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}\boldsymbol{\Sigma}_G^{\mathsf{T}}\\ \boldsymbol{\Sigma}_{Y^{a_i,n}}^{\mathsf{T}}\end{pmatrix} + \gamma_t^{a_i,n}\boldsymbol{A}_{Y^{a_i,n}}^{\mathsf{T}} = \begin{pmatrix}\alpha_D\gamma_t^{a_i,n} & \mu_n(\sigma_G^2 - \alpha_G\gamma_t^{a_i,n}) & \overline{\nu}_{a_i,n}(\sigma_G^2 - \alpha_G\gamma_t^{a_i,n})\end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G^{\mathsf{T}}\\ \boldsymbol{\Sigma}_{Y^{a_i,n}}^{\mathsf{T}}\end{pmatrix} &= \begin{pmatrix}0 & \sigma_D & 0 & 0\\ \mu_n\sigma_G & 0 & \tilde{\sigma}_{a_i,n} & w(a_i)m_{a_i,n}\\ \overline{\nu}_{a_i,n}\sigma_G & 0 & 0 & m_{a_i,n}\end{pmatrix}\begin{pmatrix}0 & \mu_n\sigma_G & \overline{\nu}_{a_i,n}\sigma_G\\ \sigma_D & 0 & 0\\ 0 & \tilde{\sigma}_{a_i,n} & 0\\ 0 & w(a_i)m_{a_i,n} & m_{a_i,n}\end{pmatrix}\\ &= \begin{pmatrix}\sigma_D^2 & 0 & 0\\ 0 & \mu_n^2\sigma_G^2 + \tilde{\sigma}_{a_i,n}^2 + w(a_i)^2m_{a_i,n}^2 & \mu_n\overline{\nu}_{a_i,n}\sigma_G^2 + w(a_i)m_{a_i,n}^2\\ 0 & \mu_n\overline{\nu}_{a_i,n}\sigma_G^2 + w(a_i)m_{a_i,n}^2 & \overline{\nu}_{a_i,n}^2\sigma_G^2 + m_{a_i,n}^2\end{pmatrix},\end{aligned}$$

so that

$$\begin{aligned}&\det\left(\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}}\right)\\ &= \sigma_D^2((\mu_n^2\sigma_G^2 + \tilde{\sigma}_{a_i,n}^2 + w(a_i)^2m_{a_i,n}^2)(\overline{\nu}_{a_i,n}^2\sigma_G^2 + m_{a_i,n}^2) - (\mu_n\overline{\nu}_{a_i,n}\sigma_G^2 + w(a_i)m_{a_i,n}^2)^2)\\ &= \sigma_D^2\left(\sigma_G^2\left(\mu_n^2 - 2\mu_n\sigma_{a_i,n}\frac{\overline{\nu}_{a_i,n}}{m_{a_i,n}} + \sigma_{\xi,n}^2\frac{\overline{\nu}_{a_i,n}^2}{m_{a_i,n}^2}\right) + \tilde{\sigma}_{a_i,n}^2\right)m_{a_i,n}^2 = \sigma_D^2(\tilde{\mu}_{a_i,n}^2\sigma_G^2 + \tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2,\end{aligned}$$

and therefore

$$\left(\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}}\right)^{-1} = \begin{pmatrix}\frac{1}{\sigma_D^2} & 0 & 0\\ 0 & \frac{\overline{\nu}_{a_i,n}^2\sigma_G^2+m_{a_i,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2} & -\frac{\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\\ 0 & -\frac{\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2} & \frac{\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\end{pmatrix}.$$

Combining everything yields

$$\begin{aligned}\boldsymbol{\Gamma}_t^{a_i,n} &= \begin{pmatrix}\alpha_D\gamma_t^{a_i,n}\\ \mu_n(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})\\ \overline{\nu}_{a_i,n}(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}\frac{1}{\sigma_D^2} & 0 & 0\\ 0 & \frac{\overline{\nu}_{a_i,n}^2\sigma_G^2+m_{a_i,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2} & -\frac{\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\\ 0 & -\frac{\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2} & \frac{\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\end{pmatrix}\\ &= \begin{pmatrix}\frac{\alpha_D\gamma_t^{a_i,n}}{\sigma_D^2} & \frac{m_{a_i,n}^2(\mu_n-w(a_i)\overline{\nu}_{a_i,n})(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2} & \frac{(\sigma_{\xi,n}^2\overline{\nu}_{a_i,n}-\mu_n w(a_i)m_{a_i,n}^2)(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\end{pmatrix},\end{aligned}$$

which proves equation (C.2) and thus concludes the proof of equation (C.1).

It remains to prove the Riccati equation (C.3). For this purpose we compute

$$\begin{aligned}&\boldsymbol{\Gamma}_t^{a_i,n}\left[\begin{pmatrix}\sigma_G & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_G & \boldsymbol{\Sigma}_{Y^{a_i,n}}\end{pmatrix}^{\mathsf{T}} + \gamma_t^{a_i,n}\boldsymbol{A}_{Y^{a_i,n}}^{\mathsf{T}}\right]^{\mathsf{T}}\\ &= \begin{pmatrix}\frac{\alpha_D\gamma_t^{a_i,n}}{\sigma_D^2}\\ \frac{m_{a_i,n}^2(\mu_n-w(a_i)\overline{\nu}_{a_i,n})(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\\ \frac{(\sigma_{\xi,n}^2\overline{\nu}_{a_i,n}-\mu_n w(a_i)m_{a_i,n}^2)(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)m_{a_i,n}^2}\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}\alpha_D\gamma_t^{a_i,n}\\ \mu_n(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})\\ \overline{\nu}_{a_i,n}(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})\end{pmatrix}\\ &= \frac{\alpha_D^2(\gamma_t^{a_i,n})^2}{\sigma_D^2} + \frac{\mu_n(\mu_n-w(a_i)\overline{\nu}_{a_i,n})(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})^2}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2} + \frac{\overline{\nu}_{a_i,n}\left(\sigma_{\xi,n}^2\frac{\overline{\nu}_{a_i,n}}{m_{a_i,n}^2}-\mu_n w(a_i)\right)(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})^2}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2}\\ &= \frac{\alpha_D^2(\gamma_t^{a_i,n})^2}{\sigma_D^2} + \frac{\mu_n^2-2\mu_n w(a_i)\overline{\nu}_{a_i,n}+\sigma_{\xi,n}^2\frac{\overline{\nu}_{a_i,n}^2}{m_{a_i,n}^2}}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2}(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})^2 = \frac{\alpha_D^2(\gamma_t^{a_i,n})^2}{\sigma_D^2} + \frac{\tilde{\mu}_{a_i,n}^2(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})^2}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2},\end{aligned}$$

which implies

$$\begin{aligned}\frac{\mathrm{d}\gamma_t^{a_i,n}}{\mathrm{d}t} &= \sigma_G^2 - 2\alpha_G\gamma_t^{a_i,n} - \left(\frac{\alpha_D^2(\gamma_t^{a_i,n})^2}{\sigma_D^2} + \frac{\tilde{\mu}_{a_i,n}^2(\sigma_G^2-\alpha_G\gamma_t^{a_i,n})^2}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2}\right)\\ &= \frac{\tilde{\sigma}_{a_i,n}^2(\sigma_G^2-2\alpha_G\gamma_t^{a_i,n})}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2} - \left(\frac{\alpha_D^2}{\sigma_D^2} + \frac{\tilde{\mu}_{a_i,n}^2\alpha_G^2}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2}\right)(\gamma_t^{a_i,n})^2,\end{aligned}$$

which is exactly equation (C.3). ∎

In a similar way we obtain the description of the public filter $\widehat{G}^n$.

Lemma C.2. *The process* $\widehat{G}^n = (\widehat{G}_t^n)_{t\geq 0}$ *defined via* $\widehat{G}_t^n := \mathrm{E}[G_t|\mathcal{G}_t^n]$, *with the filtration*

$\mathbb{G}^n = (\mathcal{G}_t^n)_{t\geq 0}$ *defined as* $\mathcal{G}_t^n := \sigma(D_s, \Xi_s^n;\, s \leq t)$*, satisfies the stochastic differential equation*

$$\mathrm{d}\widehat{G}_t^n = \alpha_G(g - \widehat{G}_t^n)\,\mathrm{d}t + \boldsymbol{\Gamma}_t^n \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^n)\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^n)\,\mathrm{d}t \end{pmatrix}, \tag{C.4}$$

with $\boldsymbol{\Gamma}_t^n = \begin{pmatrix} \Gamma_1^n(t) & \Gamma_2^n(t) \end{pmatrix} \in \mathbb{R}^{1\times 2}$ *defined via*

$$\Gamma_1^n(t) := \frac{\alpha_D \gamma_t^n}{\sigma_D^2}, \qquad \Gamma_2^n(t) := \frac{\mu_n(\sigma_G^2 - \alpha_G\gamma_t^n)}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}. \tag{C.5}$$

The mean squared error $\gamma_t^n := \mathrm{E}[(G_t - \widehat{G}_t^n)^2]$ *satisfies the Riccati equation*

$$\frac{\mathrm{d}\gamma_t^n}{\mathrm{d}t} = \frac{\sigma_{\xi,n}^2}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}(\sigma_G^2 - 2\alpha_G\gamma_t^n) - \left(\frac{\alpha_D^2}{\sigma_D^2} + \frac{\alpha_G^2\mu_n^2}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}\right)(\gamma_t^n)^2. \tag{C.6}$$

Proof. Consider the observable process

$$\mathrm{d}\underline{Y}_t^n := \begin{pmatrix} \mathrm{d}D_t + \alpha_D D_t\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G g\,\mathrm{d}t \end{pmatrix},$$

which satisfies the stochastic differential equation

$$\mathrm{d}\underline{Y}_t^n = \begin{pmatrix} \alpha_D \\ -\mu_n\alpha_G \end{pmatrix} G_t\,\mathrm{d}t + \begin{pmatrix} 0 \\ \mu_n\sigma_G \end{pmatrix} \mathrm{d}W_t^G + \begin{pmatrix} \sigma_D & 0 \\ 0 & \sigma_{\xi,n} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^D \\ \mathrm{d}W_t^{\Xi,n} \end{pmatrix}.$$

From Theorem 10.3 of Liptser and Shiryaev (2001) we thus obtain

$$\mathrm{d}\widehat{G}_t^n = \alpha_G(g - \widehat{G}_t^n)\,\mathrm{d}t + \boldsymbol{\Gamma}_t^n \left[\mathrm{d}\underline{Y}_t - \begin{pmatrix} \alpha_D \\ -\mu_n\alpha_G \end{pmatrix} \widehat{G}_t^n\,\mathrm{d}t\right],$$

with

$$\begin{aligned}\boldsymbol{\Gamma}_t^n &= \left[\begin{pmatrix} \sigma_G \\ 0 \\ 0 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 0 & \mu_n\sigma_G \\ \sigma_D & 0 \\ 0 & \sigma_{\xi,n} \end{pmatrix} + \gamma_t^n \begin{pmatrix} \alpha_D & -\mu_n\alpha_G \end{pmatrix}\right] \left[\begin{pmatrix} 0 & \sigma_D & 0 \\ \mu_n\sigma_G & 0 & \sigma_{\xi,n} \end{pmatrix} \begin{pmatrix} 0 & \mu_n\sigma_G \\ \sigma_D & 0 \\ 0 & \sigma_{\xi,n} \end{pmatrix}\right]^{-1} \\ &= \begin{pmatrix} \alpha_D\gamma_t^n & \mu_n(\sigma_G^2 - \alpha_G\gamma_t^n) \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_D^2} & 0 \\ 0 & \frac{1}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_D\gamma_t^n}{\sigma_D^2} & \frac{\mu_n(\sigma_G^2 - \alpha_G\gamma_t^n)}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2} \end{pmatrix},\end{aligned}$$

which proves equations (C.4) and (C.5). To prove equation (C.6) we use Theorem 10.3 of

Liptser and Shiryaev (2001) again, which states that

$$
\begin{aligned}
\frac{\mathrm{d}\gamma_t^n}{\mathrm{d}t} &= \sigma_G^2 - 2\alpha_G\gamma_t^n - \boldsymbol{\Gamma}_t^n \begin{pmatrix} \alpha_D\gamma_t^n \\ \mu_n(\sigma_G^2 - \alpha_G\gamma_t^n) \end{pmatrix} \\
&= \sigma_G^2 - 2\alpha_G\gamma_t^n - \frac{\alpha_D^2(\gamma_t^n)^2}{\sigma_D^2} - \frac{\mu_n^2(\sigma_G^2 - \alpha_G\gamma_t^n)^2}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2} \\
&= \frac{\sigma_{\xi,n}^2(\sigma_G^2 - 2\alpha_G\gamma_t^n)}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2} - \left(\frac{\alpha_D^2}{\sigma_D^2} + \frac{\mu_n^2\alpha_G^2}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}\right)(\gamma_t^n)^2.
\end{aligned}
$$

This concludes the proof. ■

Based on the results of Lemma C.1 and Lemma C.2 we can describe the inference wedge $\Lambda^{a_i,n} = \widehat{G}^{a_i,n} - \widehat{G}^n$.

Corollary C.3. *The process $\Lambda^{a_i,n} = (\Lambda_t^{a_i,n})_{t\geq 0}$, defined via $\Lambda_t^{a_i,n} := \widehat{G}_t^{a_i,n} - \widehat{G}_t^n$, satisfies the stochastic differential equation*

$$
\mathrm{d}\Lambda_t^{a_i,n} = -\alpha_\Lambda^n(t)\Lambda_t^{a_i,n}\,\mathrm{d}t + \begin{pmatrix} \Gamma_1^{a_i,n}(t) - \Gamma_1^n(t) \\ \Gamma_2^{a_i,n}(t) - \Gamma_2^n(t) \\ \Gamma_3^{a_i,n}(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \end{pmatrix},
$$

where $\alpha_\Lambda^n(t) := \alpha_D\Gamma_1^n(t) + \alpha_G(1 - \mu_n\Gamma_2^n(t)) = \frac{\alpha_D^2\gamma_t^n}{\sigma_D^2} + \frac{\alpha_G\sigma_{\xi,n}^2 + \mu_n^2\alpha_G^2\gamma_t^n}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}$.

Proof. By subtracting equation (C.4) from equation (C.1) we obtain

$$
\begin{aligned}
&\mathrm{d}\widehat{G}_t^{a_i,n} - \mathrm{d}\widehat{G}_t^n \\
&\quad = -\alpha_G(\widehat{G}_t^{a_i,n} - \widehat{G}_t^n)\,\mathrm{d}t + \begin{pmatrix} \Gamma_1^{a_i,n}(t) \\ \Gamma_2^{a_i,n}(t) \\ \Gamma_3^{a_i,n}(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \end{pmatrix} \\
&\qquad - \begin{pmatrix} \Gamma_1^n(t) \\ \Gamma_2^n(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^n)\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^n)\,\mathrm{d}t \end{pmatrix} \\
&\quad = -\alpha_G(\widehat{G}_t^{a_i,n} - \widehat{G}_t^n)\,\mathrm{d}t + \begin{pmatrix} \Gamma_1^{a_i,n}(t) - \Gamma_1^n(t) \\ \Gamma_2^{a_i,n}(t) - \Gamma_2^n(t) \\ \Gamma_3^{a_i,n}(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \end{pmatrix} \\
&\qquad - \begin{pmatrix} \Gamma_1^n(t) \\ \Gamma_2^n(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \alpha_D(\widehat{G}_t^{a_i,n} - \widehat{G}_t^n)\,\mathrm{d}t \\ -\mu_n\alpha_G(\widehat{G}_t^{a_i,n} - \widehat{G}_t^n)\,\mathrm{d}t \end{pmatrix},
\end{aligned}
$$

and therefore

$$\mathrm{d}\Lambda_t^{a_i,n} = -\left(\alpha_G + \mathbf{\Gamma}_t^n \begin{pmatrix} \alpha_D \\ -\mu_n\alpha_G \end{pmatrix}\right) \Lambda_t^{a_i,n}\,\mathrm{d}t + \begin{pmatrix} \Gamma_1^{a_i,n}(t) - \Gamma_1^n(t) \\ \Gamma_2^{a_i,n}(t) - \Gamma_2^n(t) \\ \Gamma_3^{a_i,n}(t) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t \end{pmatrix}.$$

By noting that

$$\begin{aligned} \alpha_G + \mathbf{\Gamma}_t^n \begin{pmatrix} \alpha_D \\ -\mu_n\alpha_G \end{pmatrix} &= \alpha_D\Gamma_1^n(t) + \alpha_G(1 - \mu_n\Gamma_2^n(t)) \\ &= \frac{\alpha_D^2\gamma_t^n}{\sigma_D^2} + \alpha_G\left(1 - \frac{\mu_n^2(\sigma_G^2 - \alpha_G\gamma_t^n)}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}\right) = \frac{\alpha_D^2\gamma_t^n}{\sigma_D^2} + \frac{\alpha_G\sigma_{\xi,n}^2 + \mu_n^2\alpha_G^2\gamma_t^n}{\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2}, \end{aligned}$$

the statement follows. ■

C.2. The filtering coefficients and their differences

Note that the steady state solutions to the Riccati equations (C.3) and (C.6) are given by

$$\gamma^{a_i,n} = \frac{\sigma_D\sqrt{(1 + \sigma_G^2\tilde{\tau}_{a_i,n}^2)(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)} - \alpha_G\sigma_D^2}{\alpha_D^2 + \tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)}, \qquad \tilde{\tau}_{a_i,n}^2 := \frac{\tilde{\mu}_{a_i,n}^2}{\tilde{\sigma}_{a_i,n}^2},$$

and

$$\gamma^n = \frac{\sigma_D\sqrt{(1 + \sigma_G^2\tau_n^2)(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)} - \alpha_G\sigma_D^2}{\alpha_D^2 + \tau_n^2(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)}, \qquad \tau_n^2 := \frac{\mu_n^2}{\sigma_{\xi,n}^2}.$$

We are now interested in the difference of the steady state mean squared errors $\gamma^n - \gamma^{a_i,n}$ between the public and the private filter.

Lemma C.4. *Let $\gamma^{a_i,n}$ and γ^n be the steady state solutions to the Riccati equations* (C.3) *and* (C.6)*, respectively. The difference $\gamma^n - \gamma^{a_i,n}$ satisfies*

$$\gamma^n - \gamma^{a_i,n} = \frac{(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1 + \sigma_G^2\tau_n^2)^{1/2} + (1 + \sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})}(\tilde{\tau}_{a_i,n}^2 - \tau_n^2).$$

Proof. Let us first note that

$$\alpha_D^2 + \tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2) - (\alpha_D^2 + \tau_n^2(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)) = (\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)(\tilde{\tau}_{a_i,n}^2 - \tau_n^2),$$

and

$$\begin{aligned}&(1+\sigma_G^2\tau_n^2)^{1/2}(\alpha_D^2+\tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))-(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}(\alpha_D^2+\tau_n^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))\\&=\alpha_D^2((1+\sigma_G^2\tau_n^2)^{1/2}-(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})\\&+(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)(\tilde{\tau}_{a_i,n}^2(1+\sigma_G^2\tau_n^2)^{1/2}-\tau_n^2(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}).\end{aligned}$$

Now, we have

$$(1+\sigma_G^2\tau_n^2)^{1/2}-(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}=\frac{\sigma_G^2(\tau_n^2-\tilde{\tau}_{a_i,n}^2)}{(1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}},$$

as well as

$$\tilde{\tau}_{a_i,n}^2(1+\sigma_G^2\tau_n^2)^{1/2}-\tau_n^2(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}=\frac{1+(1+\sigma_G^2\tau_n^2)^{1/2}(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}}{(1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}}(\tilde{\tau}_{a_i,n}^2-\tau_n^2),$$

which implies

$$\begin{aligned}&(1+\sigma_G^2\tau_n^2)^{1/2}(\alpha_D^2+\tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))-(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}(\alpha_D^2+\tau_n^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))\\&\quad=\frac{\alpha_G^2\sigma_D^2+(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)(1+\sigma_G^2\tau_n^2)^{1/2}(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}}{(1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}}(\tilde{\tau}_{a_i,n}^2-\tau_n^2).\end{aligned}$$

From these equations we eventually obtain

$$\begin{aligned}\gamma^n-\gamma^{a_i,n}&=\tfrac{-\alpha_G\sigma_D^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)(\tilde{\tau}_{a_i,n}^2-\tau_n^2)}{(\alpha_D^2+\tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))(\alpha_D^2+\tau_n^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))}\\&\quad+\tfrac{\sigma_D(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}(\alpha_G^2\sigma_D^2+(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)(1+\sigma_G^2\tau_n^2)^{1/2}(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})(\tilde{\tau}_{a_i,n}^2-\tau_n^2)}{(\alpha_D^2+\tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))(\alpha_D^2+\tau_n^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})}.\end{aligned}$$

However, note that

$$\begin{aligned}&(\alpha_G^2\sigma_D^2+(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)(1+\sigma_G^2\tau_n^2)^{1/2}(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})\\&-\alpha_G\sigma_D(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})\\&=(\alpha_G\sigma_D-(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}(1+\sigma_G^2\tau_n^2)^{1/2})(\alpha_G\sigma_D-(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2}),\end{aligned}$$

which implies

$$\begin{aligned}&\gamma^n-\gamma^{a_i,n}\\&=\tfrac{\sigma_D(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}(\alpha_G\sigma_D-(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}(1+\sigma_G^2\tau_n^2)^{1/2})(\alpha_G\sigma_D-(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})(\tilde{\tau}_{a_i,n}^2-\tau_n^2)}{(\alpha_D^2+\tilde{\tau}_{a_i,n}^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))(\alpha_D^2+\tau_n^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2))((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})},\end{aligned}$$

and therefore

$$\gamma^n - \gamma^{a_i,n} = \frac{(\alpha_G^2\sigma_D^2 + \alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2} + (1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})}(\tilde{\tau}_{a_i,n}^2 - \tau_n^2),$$

which concludes the proof. ■

To obtain a more detailed description of the difference $\gamma^n - \gamma^{a_i,n}$, we need the following result, which describes the difference $\tilde{\tau}_{a_i,n}^2 - \tau_n^2$.

Lemma C.5. *The difference $\tilde{\tau}_{a_i,n}^2 - \tau_n^2$ satisfies*

$$\tilde{\tau}_{a_i,n}^2 - \tau_n^2 = \frac{(w(a_i)\mu_n - \nu(a_i)\sigma_{\xi,n}^2)^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2}\Delta a_i.$$

Proof. By definition we have

$$\tilde{\tau}_{a_i,n}^2 - \tau_n^2 = \frac{\tilde{\mu}_{a_i,n}^2}{\tilde{\sigma}_{a_i,n}^2} - \frac{\mu_n^2}{\sigma_{\xi,n}^2} = \frac{\tilde{\mu}_{a_i,n}^2\sigma_{\xi,n}^2 - \mu_n^2\tilde{\sigma}_{a_i,n}^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2},$$

and

$$\begin{aligned}
\tilde{\mu}_{a_i,n}^2\sigma_{\xi,n}^2 - \mu_n^2\tilde{\sigma}_{a_i,n}^2 &= \left(\mu_n^2 - 2\mu_n w(a_i)\overline{\nu}_{a_i,n} + \sigma_{\xi,n}^2\frac{\overline{\nu}_{a_i,n}^2}{\Delta a_i}\right)\sigma_{\xi,n}^2 - \mu_n^2(\sigma_{\xi,n}^2 - w(a_i)^2\Delta a_i) \\
&= (\mu_n^2 w(a_i)^2 - 2\mu_n w(a_i)\nu(a_i)\sigma_{\xi,n}^2 + \nu(a_i)^2\sigma_{\xi,n}^4)\Delta a_i \\
&= (w(a_i)\mu_n - \nu(a_i)\sigma_{\xi,n}^2)^2\Delta a_i,
\end{aligned}$$

so that

$$\tilde{\tau}_{a_i,n}^2 - \tau_n^2 = \frac{(w(a_i)\mu_n - \nu(a_i)\sigma_{\xi,n}^2)^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2}\Delta a_i,$$

which concludes the proof. ■

From Lemma C.4 and Lemma C.5 we observe that $\gamma^n - \gamma^{a_i,n} \geq 0$. This makes sense because $\gamma^{a_i,n}$ refers to the mean squared error of the private filter $\widehat{G}^{a_i,n}$, while γ^n refers to the mean squared error of the public filter $\widehat{G}^n$, and the latter contains less information.

In a next step, we are interested in the differences $\Gamma_1^{a_i,n} - \Gamma_1^n$ and $\Gamma_2^{a_i,n} - \Gamma_2^n$, which appear in the stochastic differential equation of the process $\Lambda^{a_i,n}$.

Lemma C.6. *The differences $\Gamma_1^{a_i,n} - \Gamma_1^n$ and $\Gamma_2^{a_i,n} - \Gamma_2^n$ satisfy*

$$\Gamma_1^{a_i,n} - \Gamma_1^n = -\frac{\alpha_D}{\sigma_D^2} \cdot \frac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})} \cdot \frac{(w(a_i)\mu_n-\nu(a_i)\sigma_{\xi,n}^2)^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2}\Delta a_i,$$

and

$$\Gamma_2^{a_i,n} - \Gamma_2^n = \tfrac{(\mu_n\sigma_G^2\nu(a_i)+w(a_i))(\mu_n w(a_i)-\nu(a_i)\sigma_{\xi,n}^2)(\sigma_G^2-\alpha_G\gamma^n)}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)}\Delta a_i + \tfrac{\alpha_G(\mu_n-w(a_i)\overline{\nu}_{a_i,n})}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2} \cdot \tfrac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})} \cdot \tfrac{(w(a_i)\mu_n-\nu(a_i)\sigma_{\xi,n}^2)^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2}\Delta a_i.$$

Proof. By using the definitions given in Lemma C.1 and Lemma C.2 we find that

$$\begin{aligned}\Gamma_1^{a_i,n} - \Gamma_1^n &= \frac{\alpha_D\gamma^{a_i,n}}{\sigma_D^2} - \frac{\alpha_D\gamma^n}{\sigma_D^2} = \frac{\alpha_D}{\sigma_D^2}(\gamma^{a_i,n}-\gamma^n)\\ &= -\frac{\alpha_D}{\sigma_D^2}\cdot\frac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})}\cdot\frac{(w(a_i)\mu_n-\nu(a_i)\sigma_{\xi,n}^2)^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2}\Delta a_i,\end{aligned}$$

which proves the first assertion. We further have

$$\begin{aligned}&\Gamma_2^{a_i,n} - \Gamma_2^n\\ &\quad= \frac{(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\sigma_G^2-\alpha_G\gamma^{a_i,n})}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2} - \frac{\mu_n(\sigma_G^2-\alpha_G\gamma^n)}{\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2}\\ &\quad= \frac{(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)(\sigma_G^2-\alpha_G\gamma^{a_i,n}) - \mu_n(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)(\sigma_G^2-\alpha_G\gamma^n)}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)},\end{aligned}$$

which together with

$$\begin{aligned}&(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)(\sigma_G^2-\alpha_G\gamma^{a_i,n}) - \mu_n(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)(\sigma_G^2-\alpha_G\gamma^n)\\ &= ((\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2) - \mu_n(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2))(\sigma_G^2-\alpha_G\gamma^n)\\ &\quad - \alpha_G(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)(\gamma^{a_i,n}-\gamma^n)\\ &= (\mu_n\sigma_G^2(\mu_n^2-\tilde{\mu}_{a_i,n}^2) + \mu_n(\sigma_{\xi,n}^2-\tilde{\sigma}_{a_i,n}^2) - w(a_i)\overline{\nu}_{a_i,n}(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2))(\sigma_G^2-\alpha_G\gamma^n)\\ &\quad + \frac{\alpha_G(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})}(\tilde{\tau}_{a_i,n}^2-\tau_n^2)\\ &= (\mu_n\sigma_G^2\nu(a_i)+w(a_i))(\mu_n w(a_i)-\nu(a_i)\sigma_{\xi,n}^2)(\sigma_G^2-\alpha_G\gamma^n)\Delta a_i\\ &\quad + \frac{\alpha_G(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})}(\tilde{\tau}_{a_i,n}^2-\tau_n^2)\end{aligned}$$

yields

$$\Gamma_2^{a_i,n} - \Gamma_2^n = \tfrac{(\mu_n\sigma_G^2\nu(a_i)+w(a_i))(\mu_n w(a_i)-\nu(a_i)\sigma_{\xi,n}^2)(\sigma_G^2-\alpha_G\gamma^n)}{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)(\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2)}\Delta a_i + \tfrac{\alpha_G(\mu_n-w(a_i)\overline{\nu}_{a_i,n})}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2} \cdot \tfrac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma^{a_i,n}}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tilde{\tau}_{a_i,n}^2)^{1/2})} \cdot \tfrac{(w(a_i)\mu_n-\nu(a_i)\sigma_{\xi,n}^2)^2}{\tilde{\sigma}_{a_i,n}^2\sigma_{\xi,n}^2}\Delta a_i.$$

This concludes the proof. ∎

C.3. Convergence of discrete-economy filtering equations

We are now interested in the convergence of the public and the private filters $\widehat{G}^n$ and $\widehat{G}^{a_i,n}$ in the discrete economy as $n \to \infty$. In this section we prove that both discrete-economy filters, $\widehat{G}^n$ and $\widehat{G}^{a_i,n}$, converge to the continuous-economy public filter $\widehat{G} = (\widehat{G}_t)_{t\geq 0}$, which satisfies the stochastic differential equation

$$\mathrm{d}\widehat{G}_t = \alpha_G(g - \widehat{G}_t)\,\mathrm{d}t + \begin{pmatrix} \frac{\alpha_D\gamma_t}{\sigma_D^2} \\ \frac{\mu(\sigma_G^2 - \alpha_G\gamma_t)}{\mu^2\sigma_G^2+\sigma_\xi^2} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t)\,\mathrm{d}t \\ \mathrm{d}\Xi_t - \mu\alpha_G(g - \widehat{G}_t)\,\mathrm{d}t \end{pmatrix},$$

where the mean-squared error $\gamma_t \coloneqq \mathrm{E}[(G_t - \widehat{G}_t)^2]$ satisfies the Riccati equation

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = \frac{\sigma_\xi^2}{\mu^2\sigma_G^2 + \sigma_\xi^2}(\sigma_G^2 - 2\alpha_G\gamma_t) - \left(\frac{\alpha_D^2}{\sigma_D^2} + \frac{\alpha_G^2\mu^2}{\mu^2\sigma_G^2 + \sigma_\xi^2}\right)\gamma_t^2. \tag{C.7}$$

For simplicity, we use the familiar notation $\mathbf{\Gamma}_t = \begin{pmatrix}\Gamma_1(t) & \Gamma_2(t)\end{pmatrix} \in \mathbb{R}^{1\times 2}$ defined via

$$\Gamma_1(t) \coloneqq \frac{\alpha_D\gamma_t}{\sigma_D^2}, \qquad \Gamma_2(t) \coloneqq \frac{\mu(\sigma_G^2 - \alpha_G\gamma_t)}{\mu^2\sigma_G^2 + \sigma_\xi^2}.$$

As before, since for our purpose it is enough to consider the steady state versions we focus on this case.

Lemma C.7. *The discrete-economy public filter $\widehat{G}^n$ converges uniformly on compacts in probability to the continuous-economy public filter $\widehat{G} = (\widehat{G}_t)_{t\geq 0}$.*

Proof. Let us first note that

$$\begin{aligned}
\mathrm{d}\widehat{G}_t - \mathrm{d}\widehat{G}_t^n &= -\alpha_G(\widehat{G}_t - \widehat{G}_t^n)\,\mathrm{d}t + \begin{pmatrix}\Gamma_1 \\ \Gamma_2\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t)\,\mathrm{d}t \\ \mathrm{d}\Xi_t - \mu\alpha_G(g - \widehat{G}_t)\,\mathrm{d}t \end{pmatrix} \\
&\quad - \begin{pmatrix}\Gamma_1^n \\ \Gamma_2^n\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^n)\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n\alpha_G(g - \widehat{G}_t^n)\,\mathrm{d}t \end{pmatrix} \\
&= -\alpha_G(\widehat{G}_t - \widehat{G}_t^n)\,\mathrm{d}t + \begin{pmatrix}\Gamma_1 - \Gamma_1^n \\ \Gamma_2 - \Gamma_2^n\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t)\,\mathrm{d}t \\ \mathrm{d}\Xi_t - \mu\alpha_G(g - \widehat{G}_t)\,\mathrm{d}t \end{pmatrix} \\
&\quad - \begin{pmatrix}\Gamma_1^n \\ \Gamma_2^n\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \alpha_D(\widehat{G}_t - \widehat{G}_t^n)\,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mathrm{d}\Xi_t + (\mu - \mu_n)\alpha_G(g - \widehat{G}_t)\,\mathrm{d}t - \alpha_G\mu_n(\widehat{G}_t - \widehat{G}_t^n)\,\mathrm{d}t \end{pmatrix}.
\end{aligned}$$

Hence, by using the definition $\alpha_\Lambda^n = \alpha_G + \Gamma_1^n \alpha_D - \alpha_G \mu_n \Gamma_2^n$ (see Corollary C.3) we may write

$$\begin{aligned} \mathrm{d}(\widehat{G}_t - \widehat{G}_t^n) &= -\alpha_\Lambda^n (\widehat{G}_t - \widehat{G}_t^n)\,\mathrm{d}t + \begin{pmatrix} \Gamma_1 - \Gamma_1^n \\ \Gamma_2 - \Gamma_2^n \end{pmatrix}^\mathsf{T} \boldsymbol{Q}\,\mathrm{d}\underline{\widehat{W}}_t \\ &\quad - \Gamma_2^n \big(\mathrm{d}\Xi_t^n - \mathrm{d}\Xi_t + (\mu - \mu_n)\alpha_G (g - \widehat{G}_t)\,\mathrm{d}t \big), \end{aligned}$$

where $\boldsymbol{Q} \in \mathbb{R}^{2\times 2}$ and $\underline{\widehat{W}}$ are obtained via Lemma C.11 and satisfy

$$\boldsymbol{Q}\,\mathrm{d}\underline{\widehat{W}}_t = \begin{pmatrix} \mathrm{d}D_t + \alpha_D (D_t - \widehat{G}_t)\,\mathrm{d}t \\ \mathrm{d}\Xi_t - \mu\alpha_G (g - \widehat{G}_t)\,\mathrm{d}t \end{pmatrix}, \qquad \boldsymbol{Q}\boldsymbol{Q}^\mathsf{T} = \begin{pmatrix} \sigma_D^2 & 0 \\ 0 & \mu^2\sigma_G^2 + \sigma_\xi^2 \end{pmatrix}.$$

By applying Itô's formula we thus obtain

$$\begin{aligned} \widehat{G}_t - \widehat{G}_t^n &= (\widehat{G}_0 - \widehat{G}_0^n)\,\mathrm{e}^{-\alpha_\Lambda^n t} + \begin{pmatrix} \Gamma_1 - \Gamma_1^n \\ \Gamma_2 - \Gamma_2^n \end{pmatrix}^\mathsf{T} \boldsymbol{Q} \int_0^t \mathrm{e}^{-\alpha_\Lambda^n (t-s)}\,\mathrm{d}\underline{\widehat{W}}_s \\ &\quad - \Gamma_2^n \int_0^t \mathrm{e}^{-\alpha_\Lambda^n (t-s)} \big(\mathrm{d}\Xi_s^n - \mathrm{d}\Xi_s + (\mu - \mu_n)\alpha_G (g - \widehat{G}_s)\,\mathrm{d}s \big). \end{aligned}$$

Moreover, we may also note that

$$\mathrm{d}\Xi_t^n - \mathrm{d}\Xi_t = (\mu_n - \mu)\,\mathrm{d}G_t + \sigma_{\xi,n}\,\mathrm{d}W_t^{\Xi,n} - \sigma_\xi\,\mathrm{d}W_t^\Xi,$$

so that

$$\begin{aligned} &\mathrm{d}\Xi_t^n - \mathrm{d}\Xi_t + (\mu - \mu_n)\alpha_G (g - \widehat{G}_t)\,\mathrm{d}t \\ &\quad = -(\mu_n - \mu)(G_t - \widehat{G}_t)\,\mathrm{d}t + (\mu_n - \mu)\sigma_G\,\mathrm{d}W_t^G + \sigma_{\xi,n}\,\mathrm{d}W_t^{\Xi,n} - \sigma_\xi\,\mathrm{d}W_t^\Xi. \end{aligned}$$

Now, Doob's L^2 maximal inequality yields

$$\begin{aligned} &\mathrm{E}\left[\sup_{0\le t\le T} \left| \begin{pmatrix} \Gamma_1 - \Gamma_1^n \\ \Gamma_2 - \Gamma_2^n \end{pmatrix}^\mathsf{T} \boldsymbol{Q} \int_0^t \mathrm{e}^{-\alpha_\Lambda^n (t-s)}\,\mathrm{d}\underline{\widehat{W}}_s \right| \right] \\ &\quad \le 2\,\mathrm{E}\left[\left| \begin{pmatrix} \Gamma_1 - \Gamma_1^n \\ \Gamma_2 - \Gamma_2^n \end{pmatrix}^\mathsf{T} \boldsymbol{Q} \int_0^T \mathrm{e}^{-\alpha_\Lambda^n (t-s)}\,\mathrm{d}\underline{\widehat{W}}_s \right|^2 \right]^{1/2} \\ &\quad = 2 \left((\Gamma_1 - \Gamma_1^n)^2 \sigma_D^2 + (\Gamma_2 - \Gamma_2^n)^2 (\mu^2\sigma_G^2 + \sigma_\xi^2) \right) \int_0^T \mathrm{e}^{-2\alpha_\Lambda^n (t-s)}\,\mathrm{d}s \Bigg)^{1/2}, \end{aligned}$$

and similarly

$$\mathrm{E}\left[\sup_{0\le t\le T}\left|\Gamma_2^n(\mu_n-\mu)\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\sigma_G\,\mathrm{d}W_s^G\right|\right] \le 2\left((\Gamma_2^n)^2(\mu_n-\mu)^2\sigma_G^2\int_0^T \mathrm{e}^{-2\alpha_\Lambda^n(t-s)}\,\mathrm{d}s\right)^{1/2},$$

as well as

$$\begin{aligned}&\mathrm{E}\left[\sup_{0\le t\le T}\left|\Gamma_2^n\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\mathrm{d}\big(\sigma_{\xi,n}W_s^{\Xi,n}-\sigma_\xi W_s^{\Xi}\big)\right|\right]\\ &\qquad\le 2\left((\Gamma_2^n)^2\int_0^1 (w^n(a)-w(a))^2\,\mathrm{d}a\int_0^T \mathrm{e}^{-2\alpha_\Lambda^n(t-s)}\,\mathrm{d}s\right)^{1/2}.\end{aligned}$$

Note that the last inequality follows from the fact that

$$\sigma_{\xi,n}W_t^{\Xi,n}-\sigma_\xi W_t^{\Xi} = \int_{[(0,0),(1,t))}(w^n(a)-w(a))\,\mathrm{d}W_{(a,s)}^{\xi},$$

which is a scaled-Brownian motion in t-direction with scaling factor

$$\left(\int_0^1 (w^n(a)-w(a))^2\,\mathrm{d}a\right)^{1/2} = \left(\sigma_{\xi,n}^2-2\int_0^1 w^n(a)w(a)\,\mathrm{d}a+\sigma_\xi^2\right)^{1/2}.$$

Moreover, note that

$$\begin{aligned}&\mathrm{E}\left[\sup_{0\le t\le T}\left|\Gamma_2^n\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}(\mu_n-\mu)(G_s-\widehat{G}_s)\,\mathrm{d}s\right|\right]\\ &\qquad\le |\Gamma_2^n|\cdot|\mu_n-\mu|\int_0^T \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\mathrm{E}[|G_s-\widehat{G}_s|]\,\mathrm{d}s \le |\Gamma_2^n|\cdot|\mu_n-\mu|\gamma^{1/2}\int_0^T \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\mathrm{d}s,\end{aligned}$$

where we use that $\mathrm{E}[|G_s-\widehat{G}_s|]\le \mathrm{E}[|G_s-\widehat{G}_s|^2]^{1/2}=\gamma_s\le \lim_{t\to\infty}\gamma_t=\gamma$. Finally, note that the Brownian sheet is zero whenever one of its coordinates is zero, which implies

$$\sup_{0\le t\le T}|\widehat{G}_0-\widehat{G}_0^n|\,\mathrm{e}^{-\alpha_\Lambda^n t} = |\widehat{G}_0-\widehat{G}_0^n| = |\mu-\mu_n|\cdot|G_0|.$$

To conclude the proof, it is therefore enough to show that $\mu_n\to\mu$ and $\sigma_{\xi,n}\to\sigma_\xi$, which then implies $\Gamma_1^n\to\Gamma_1$ and $\Gamma_2^n\to\Gamma_2$. For this purpose, let us first note that the difference of mean-squared errors $\gamma-\gamma^n$ satisfies

$$\gamma^n-\gamma = \frac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^n\gamma}{\sigma_D((1+\sigma_G^2\tau_n^2)^{1/2}+(1+\sigma_G^2\tau^2)^{1/2})}(\tau^2-\tau_n^2), \qquad \tau^2=\frac{\mu^2}{\sigma_\xi^2}.$$

This identity is obtained as a corollary of Lemma C.4. However, since

$$\lim_{n\to\infty}\mu_n = \lim_{n\to\infty}\int_0^1 \nu^n(a)w^n(a)\,\mathrm{d}a = \int_0^1 \nu(a)w(a)\,\mathrm{d}a = \mu,$$

as well as

$$\lim_{n\to\infty} \sigma_{\xi,n}^2 = \lim_{n\to\infty} \int_0^1 w^n(a)^2 \,\mathrm{d}a = \int_0^1 w(a)^2 \,\mathrm{d}a = \sigma_\xi^2,$$

we obtain $\lim_{n\to\infty} \tau_n^2 = \tau^2$ and therefore $\lim_{n\to\infty}(\gamma^n - \gamma) = 0$. Since $\Gamma_1 - \Gamma_1^n = \frac{\alpha_D}{\sigma_D^2}(\gamma - \gamma^n)$ we thus obtain $\lim_{n\to\infty}(\Gamma_1 - \Gamma_1^n) = 0$. Furthermore, it holds that

$$\lim_{n\to\infty} (\Gamma_2 - \Gamma_2^n) = \lim_{n\to\infty} \left(\frac{\mu(\sigma_G^2 - \alpha_G \gamma)}{\mu^2 \sigma_G^2 + \sigma_\xi^2} - \frac{\mu_n(\sigma_G^2 - \alpha_G \gamma^n)}{\mu_n^2 \sigma_G^2 + \sigma_{\xi,n}^2} \right) = 0.$$

Together with the above we eventually find that

$$\mathrm{E}\left[1 \wedge \sup_{0\le t\le T} |\widehat{G}_t^n - \widehat{G}_t|\right] \to 0 \qquad n \to \infty,$$

which is enough to conclude that $\widehat{G}_t^n \to \widehat{G}_t$ uniformly on compacts in probability. ■

We now obtain the convergence of the discrete-economy private filters $\widehat{G}^{a_i,n}$ to the continuous-economy public filter $\widehat{G}$ as a corollary of Lemma C.7.

Corollary C.8. *The discrete-economy private filter $\widehat{G}^{a_i,n}$ converges uniformly on compacts in probability to the continuous-economy public filter $\widehat{G} = (\widehat{G}_t)_{t\ge 0}$.*

Proof. From Corollary C.3 we know that $\Lambda^{a_i,n} = (\widehat{G}^{a_i,n} - \widehat{G}^n)$ satisfies the stochastic differential equation

$$\mathrm{d}\Lambda_t^{a_i,n} = -\alpha_\Lambda^n \Lambda_t^{a_i,n} \,\mathrm{d}t + \begin{pmatrix} \Gamma_1^{a_i,n} - \Gamma_1^n \\ \Gamma_2^{a_i,n} - \Gamma_2^n \\ \Gamma_3^{a_i,n} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_t + \alpha_D(D_t - \widehat{G}_t^{a_i,n}) \,\mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n \alpha_G(g - \widehat{G}_t^{a_i,n}) \,\mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n} \alpha_G (g - \widehat{G}_t^{a_i,n}) \,\mathrm{d}t \end{pmatrix}$$

in the steady state. By Itô's formula we thus obtain

$$\Lambda_t^{a_i,n} = \Lambda_0^{a_i,n} \,\mathrm{e}^{-\alpha_\Lambda^n t} + \begin{pmatrix} \Gamma_1^{a_i,n} - \Gamma_1^n \\ \Gamma_2^{a_i,n} - \Gamma_2^n \\ \Gamma_3^{a_i,n} \end{pmatrix}^{\mathsf{T}} \int_0^t \mathrm{e}^{-\alpha_\Lambda^n (t-s)} \,\boldsymbol{Q}^{a_i,n} \,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n},$$

and so again by Doob's L^2 maximal inequality we find that

$$\mathrm{E}\left[\sup_{0\le t\le T} \left| \begin{pmatrix} \Gamma_1^{a_i,n} - \Gamma_1^n \\ \Gamma_2^{a_i,n} - \Gamma_2^n \\ \Gamma_3^{a_i,n} \end{pmatrix}^{\mathsf{T}} \int_0^t \mathrm{e}^{-\alpha_\Lambda^n (t-s)} \,\boldsymbol{Q}^{a_i,n} \,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n} \right| \right] \le 2\eta_{\Lambda\Lambda}^{a_i,n} \int_0^T \mathrm{e}^{-2\alpha_\Lambda^n (t-s)} \,\mathrm{d}s.$$

On the other hand, again by the fact that the Brownian sheet is zero whenever one coordinate is zero, we find that

$$\sup_{0\le t\le T} |\Lambda_0^{a_i,n}|\,\mathrm{e}^{-\alpha_\Lambda^n t} = |\Lambda_0^{a_i,n}| = 0.$$

Since $\lim_{n\to\infty} \eta_{\Lambda\Lambda}^{a_i,n} \to 0$ it follows that $\Lambda_t^{a_i,n} \to 0$ uniformly on compacts in probability. The result now follows together with Lemma C.7. ■

C.4. The limit of aggregated filtering increments

We are now ready to prove the main result of this section. In particular, we can now describe the limit

$$(\varphi * \Lambda)_{(a,t)} \coloneqq \lim_{n\to\infty} \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)\Lambda_t^{a_i,n}, \tag{C.8}$$

for some function $\varphi\colon [0,1) \to \mathbb{R}$. The notation $(\varphi * \Lambda)_{(a,t)}$ shall indicate that we are considering a stochastic-integral-type object in agent direction of φ with respect to the filtering difference.

Theorem C.9. *Let $(\varphi*\Lambda)$ be defined as in* (C.8)*. The convergence is uniformly on compacts in probability, and the process $(\varphi * \Lambda)$ satisfies*

$$\begin{aligned}(\varphi * \Lambda)_{(a,t)} + \alpha_\Lambda \int_0^t (\varphi * \Lambda)_{(a,s)}\,\mathrm{d}s &= \int_0^a \varphi(u)(w(u)\mu - \nu(u)\sigma_\xi^2)^2\,\mathrm{d}u \int_0^t \mathrm{d}f_1(D_s, \widehat{G}_s, \Xi_s)\\ &\quad + \int_{[(0,0),(a,t))} \varphi(u)(\nu(u)\sigma_\xi^2 - w(u)\mu)\,\mathrm{d}f_2(\xi_{(u,s)}, \widehat{G}_s, \Xi_s),\end{aligned}$$

where

$$\mathrm{d}f_1(D_s, \widehat{G}_s, \Xi_s) = \tfrac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2} \begin{pmatrix} -\frac{\alpha_D\gamma}{2\sigma_D^2\sigma_\xi^2} \\ \frac{\alpha_G\mu\gamma}{2\sigma_\xi^2(\mu^2\sigma_G^2+\sigma_\xi^2)} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_s + \alpha_D(D_s - \widehat{G}_s)\,\mathrm{d}s \\ \mathrm{d}\Xi_s - \mu\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s \end{pmatrix},$$

$$\mathrm{d}f_2(\xi_{(u,s)}, \widehat{G}_s, \Xi_s) = \begin{pmatrix} -\frac{(\sigma_G^2-\alpha_G\gamma)(\mu\sigma_G^2\nu(u)+w(u))}{(\mu^2\sigma_G^2+\sigma_\xi^2)^2} \\ \frac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}\Xi_s\,\mathrm{d}u - \mu\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u \\ \mathrm{d}\xi_{(u,s)} - \nu(u)\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u \end{pmatrix}.$$

Proof. From Corollary C.3 we first obtain

$$\begin{aligned}\sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)\Lambda_t^{a_i,n} &= \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)\Lambda_0^{a_i,n} - \alpha_\Lambda^n \int_0^t \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)\Lambda_s^{a_i,n}\,\mathrm{d}s\\ &\quad + \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i) \begin{pmatrix} \Gamma_1^{a_i,n} - \Gamma_1^n \\ \Gamma_2^{a_i,n} - \Gamma_2^n \\ \Gamma_3^{a_i,n} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} D_t - D_0 + \alpha_D \int_0^t (D_s - \widehat{G}_s^{a_i,n})\,\mathrm{d}s \\ \Xi_t^n - \Xi_0^n - \mu_n\alpha_G \int_0^t (g - \widehat{G}_s^{a_i,n})\,\mathrm{d}s \\ \zeta_t^{a_i,n} - \zeta_0^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G \int_0^t (g - \widehat{G}_s^{a_i,n})\,\mathrm{d}s \end{pmatrix}.\end{aligned}$$

Now, from Lemma C.6 and Corollary C.8, together with Theorem IV.2.12 of Revuz and Yor (1999) it holds

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)(\Gamma_1^{a_i,n}-\Gamma_1^n)\left(D_t - D_0 + \alpha_D \int_0^t (D_s - \widehat{G}_s^{a_i,n})\,\mathrm{d}s\right)\\
&= -\tfrac{\alpha_D}{\sigma_D^2}\cdot \tfrac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4} \int_0^a \varphi(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2\,\mathrm{d}u\big(D_t - D_0 + \alpha_D \textstyle\int_0^t (D_s-\widehat{G}_s)\,\mathrm{d}s\big)\\
&= -\tfrac{\alpha_D}{\sigma_D^2}\cdot \tfrac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4} \int_{[(0,0),(a,t))} \varphi(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2\,\mathrm{d}u(\mathrm{d}D_s + \alpha_D(D_s-\widehat{G}_s)\,\mathrm{d}s),
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)(\Gamma_2^{a_i,n}-\Gamma_2^n)\left(\Xi_t^n - \Xi_0^n - \mu_n\alpha_G \int_0^t (g - \widehat{G}_s^{a_i,n})\,\mathrm{d}s\right)\\
&= \tfrac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2} \int_0^a \tfrac{\varphi(u)(\mu\sigma_G^2\nu(u)+w(u))(\mu w(u)-\nu(u)\sigma_\xi^2)}{\mu^2\sigma_G^2+\sigma_\xi^2}\,\mathrm{d}u\big(\Xi_t - \mu\alpha_G\textstyle\int_0^t(g-\widehat{G}_s)\,\mathrm{d}s\big)\\
&+ \tfrac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4} \int_0^a \tfrac{\alpha_G\mu\varphi(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2}{\mu^2\sigma_G^2+\sigma_\xi^2}\,\mathrm{d}u\big(\Xi_t - \mu\alpha_G\textstyle\int_0^t(g-\widehat{G}_s)\,\mathrm{d}s\big)\\
&= \tfrac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2} \int_{[(0,0),(a,t))} \tfrac{\varphi(u)(\mu\sigma_G^2\nu(u)+w(u))(\mu w(u)-\nu(u)\sigma_\xi^2)}{\mu^2\sigma_G^2+\sigma_\xi^2}\,\mathrm{d}u(\mathrm{d}\Xi_s - \mu\alpha_G(g-\widehat{G}_s)\,\mathrm{d}s)\\
&+ \tfrac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4} \int_{[(0,0),(a,t))} \tfrac{\alpha_G\mu\varphi(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2}{\mu^2\sigma_G^2+\sigma_\xi^2}\,\mathrm{d}u(\mathrm{d}\Xi_s - \mu\alpha_G(g-\widehat{G}_s)\,\mathrm{d}s),
\end{aligned}
$$

uniformly on compacts in probability. Here, we use that

$$
\Xi_0 = \left(\mu(G_t - G_0) + \int_{[(0,0),(1,t))} w(a)\,\mathrm{d}W_{(a,s)}^\xi\right)\Bigg|_{t=0} = \int_{[(0,0),(1,0))} w(a)\,\mathrm{d}W_{(a,s)}^\xi = 0,
$$

which is a direct consequence of the Brownian sheet being equal to zero on the boundary. Further, it holds that

$$
\begin{aligned}
&\sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)\Gamma_3^{a_i,n}\left(\zeta_t^{a_i,n} - \zeta_0^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G \int_0^t (g - \widehat{G}_s^{a_i,n})\,\mathrm{d}s\right)\\
&= \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \tfrac{\varphi(a_i)(\nu(a_i)\sigma_{\xi,n}^2 - w(a_i)\mu_n)(\sigma_G^2-\alpha_G\gamma^{a_i,n})\left(\Delta_1\xi_{(a_i,t)}^n - \Delta_1\xi_{(a_i,0)}^n - \nu(a_i)\Delta a_i\alpha_G\int_0^t(g-\widehat{G}_s^{a_i,n})\,\mathrm{d}s\right)}{\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2},
\end{aligned}
$$

which by Corollary C.8 and Theorem IV.2.12 of Revuz and Yor (1999) yields

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \varphi(a_i)\Gamma_3^{a_i,n}\left(\zeta_t^{a_i,n} - \zeta_0^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G \int_0^t (g-\widehat{G}_s^{a_i,n})\,\mathrm{d}s\right)\\
&\quad = \tfrac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2} \int_0^a \varphi(u)(\nu(u)\sigma_\xi^2 - w(u)\mu)(\partial_1\xi_{(u,1)} - \partial_1\xi_{(u,0)} - \nu(u)\alpha_G \textstyle\int_0^t (g-\widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u)\\
&\quad = \tfrac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2} \int_{[(0,0),(a,t))} \varphi(u)(\nu(u)\sigma_\xi^2 - w(u)\mu)(\mathrm{d}\xi_{(u,s)} - \nu(u)\alpha_G(g-\widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u),
\end{aligned}$$

uniformly on compacts in probability. Now, it is worth noting that

$$\begin{aligned}
\sigma_G^2 - \alpha_G\gamma &= \sigma_G^2 - \tfrac{\alpha_G\sigma_D\sqrt{(1+\sigma_G^2\tau^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}-\alpha_G^2\sigma_D^2}{\alpha_D^2+\tau^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}\\
&= \tfrac{(1+\sigma_G^2\tau^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)-\alpha_G\sigma_D\sqrt{(1+\sigma_G^2\tau^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}}{\alpha_D^2+\tau^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}\\
&= \tfrac{\sqrt{(1+\sigma_G^2\tau^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}}{\sigma_D} \cdot \tfrac{\sigma_D\sqrt{(1+\sigma_G^2\tau^2)(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}-\alpha_G\sigma_D^2}{\alpha_D^2+\tau^2(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)}\\
&= \tfrac{(1+\sigma_G^2\tau^2)^{1/2}(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma}{\sigma_D},
\end{aligned}$$

which implies

$$\frac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{\gamma}{2\sigma_\xi^2} = \frac{\sigma_G^2-\alpha_G\gamma}{1+\sigma_G^2\tau^2}\cdot\frac{\gamma}{2\sigma_\xi^4} = \frac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4},$$

and eventually

$$\frac{\alpha_D}{\sigma_D^2}\cdot\frac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4} = \frac{\alpha_D}{\sigma_D^2}\cdot\frac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{\gamma}{2\sigma_\xi^2} = \frac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{\Gamma_1}{2\sigma_\xi^2},$$

as well as

$$\frac{\alpha_G\mu}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{(\alpha_G^2\sigma_D^2+\alpha_D^2\sigma_G^2)^{1/2}\gamma^2}{2\sigma_D(1+\sigma_G^2\tau^2)^{1/2}\sigma_\xi^4} = \frac{\alpha_G\mu}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{\sigma_G^2-\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{\gamma}{2\sigma_\xi^2} = \frac{\alpha_G\gamma}{\mu^2\sigma_G^2+\sigma_\xi^2}\cdot\frac{\Gamma_2}{2\sigma_\xi^2}.$$

Since $\lim_{n\to\infty} \alpha_\Lambda^n = \alpha_D\Gamma_1 + \alpha_G(1-\mu\Gamma_2) = \alpha_\Lambda$, we therefore obtain

$$\begin{aligned}
(\varphi * \Lambda)_{(a,t)} &= (\varphi * \Lambda)_{(a,0)} - \alpha_\Lambda \int_0^t (\varphi * \Lambda)_{(a,s)}\,\mathrm{d}s \\
&\quad - \tfrac{\sigma_G^2 - \alpha_G\gamma}{\mu^2\sigma_G^2 + \sigma_\xi^2} \int_{[(0,0),(a,t))} \tfrac{\alpha_D\gamma}{2\sigma_D^2\sigma_\xi^2} \varphi(u)(w(u)\mu - \nu(u)\sigma_\xi^2)^2 (\mathrm{d}D_s\,\mathrm{d}u + \alpha_D(D_s - \widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u) \\
&\quad + \tfrac{\sigma_G^2 - \alpha_G\gamma}{\mu^2\sigma_G^2 + \sigma_\xi^2} \int_{[(0,0),(a,t))} \tfrac{\varphi(u)(\mu\sigma_G^2\nu(u) + w(u))(\mu w(u) - \nu(u)\sigma_\xi^2)}{\mu^2\sigma_G^2 + \sigma_\xi^2} (\mathrm{d}\Xi_s\,\mathrm{d}u - \mu\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u) \\
&\quad + \tfrac{\sigma_G^2 - \alpha_G\gamma}{\mu^2\sigma_G^2 + \sigma_\xi^2} \int_{[(0,0),(a,t))} \tfrac{\alpha_G\mu\gamma}{2\sigma_\xi^2} \cdot \tfrac{\varphi(u)(w(u)\mu - \nu(u)\sigma_\xi^2)^2}{\mu^2\sigma_G^2 + \sigma_\xi^2} (\mathrm{d}\Xi_s\,\mathrm{d}u - \mu\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u) \\
&\quad + \tfrac{\sigma_G^2 - \alpha_G\gamma}{\mu^2\sigma_G^2 + \sigma_\xi^2} \int_{[(0,0),(a,t))} \varphi(u)(\nu(u)\sigma_\xi^2 - w(u)\mu)(\mathrm{d}\xi_{(u,s)} - \nu(u)\alpha_G(g - \widehat{G}_s)\,\mathrm{d}s\,\mathrm{d}u).
\end{aligned}$$

The proof follows by noting that $\zeta_0^{a_i,n} = (\overline{\nu}_{a_i,n}(G_t - G_0) + \Delta_1 W^\xi_{(a_i,t)})|_{t=0} = \Delta_1 W^\xi_{(a_i,0)} = 0$, and therefore $\Lambda_0^{a_i,n} = 0$ for all $0 \le i \le n-1$ for every $n \ge 2$. The stated form then follows by collecting the two terms carrying $\varphi(u)(w(u)\mu - \nu(u)\sigma_\xi^2)^2$ into $\mathrm{d}f_1$, and the two terms carrying $\varphi(u)(\nu(u)\sigma_\xi^2 - w(u)\mu)$ into $\mathrm{d}f_2$. ■

C.5. The filter as an Ornstein-Uhlenbeck process

We conclude this section by providing an alternative way to describe the filtering equations. In particular, the goal is to write the filtering equations from Lemma C.1, Lemma C.2 and Corollary C.3 as Ornstein-Uhlenbeck processes, whose Brownian motion is adapted to the filtration generated by the observable processes. For completeness we start by restating the result on optimal linear non-stationary filtering based on the work of Kalman and Bucy (1961). The following result is from Liptser and Shiryaev (2001) (see Theorem 10.3) with adjusted notation.

Theorem C.10. *Suppose that the unobservable process $\underline{X} = (\underline{X}(t))_{t\ge 0}$ satisfies the stochastic differential equation*

$$\mathrm{d}\underline{X}(t) = [\underline{b}_X(t) + \boldsymbol{A}_{XX}(t)\underline{X}(t) + \boldsymbol{A}_{XY}(t)\underline{Y}(t)]\,\mathrm{d}t + \boldsymbol{\Sigma}_{XX}(t)\,\mathrm{d}\underline{W}^X(t) + \boldsymbol{\Sigma}_{XY}(t)\,\mathrm{d}\underline{W}^Y(t),$$

while the observable process $\underline{Y} = (\underline{Y}(t))_{t\ge 0}$ satisfies the stochastic differential equation

$$\mathrm{d}\underline{Y}(t) = [\underline{b}_Y(t) + \boldsymbol{A}_{YX}(t)\underline{X}(t) + \boldsymbol{A}_{YY}(t)\underline{Y}(t)]\,\mathrm{d}t + \boldsymbol{\Sigma}_{YX}(t)\,\mathrm{d}\underline{W}^X(t) + \boldsymbol{\Sigma}_{YY}(t)\,\mathrm{d}\underline{W}^Y(t),$$

where $\underline{W}^X$ and $\underline{W}^Y$ are independent Brownian motions with values in $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively, for some $k, l \in \mathbb{N}$, and where the vectors $\underline{b}_X(t) \in \mathbb{R}^k$ and $\underline{b}_Y(t) \in \mathbb{R}^l$ satisfy

$$\int_0^T (\|\underline{b}_X(t)\|_1 + \|\underline{b}_Y(t)\|_2^2)\,\mathrm{d}t < \infty,$$

the matrices $\boldsymbol{A}_{XX}(t) \in \mathbb{R}^{k\times k}$, $\boldsymbol{A}_{XY}(t) \in \mathbb{R}^{k\times l}$, $\boldsymbol{A}_{YX}(t) \in \mathbb{R}^{l\times k}$ *and* $\boldsymbol{A}_{YY}(t) \in \mathbb{R}^{l\times l}$ *satisfy*

$$\int_0^T (\|\boldsymbol{A}_{XX}(t)\|_1 + \|\boldsymbol{A}_{XY}(t)\|_1)\,\mathrm{d}t < \infty, \qquad \int_0^T (\|\boldsymbol{A}_{YX}(t)\|_2^2 + \|\boldsymbol{A}_{YY}(t)\|_2^2)\,\mathrm{d}t < \infty,$$

and where the matrices $\boldsymbol{\Sigma}_{XX}(t) \in \mathbb{R}^{k\times k}$, $\boldsymbol{\Sigma}_{XY}(t) \in \mathbb{R}^{k\times l}$, $\boldsymbol{\Sigma}_{YX}(t) \in \mathbb{R}^{l\times k}$ *and* $\boldsymbol{\Sigma}_{YY}(t) \in \mathbb{R}^{l\times l}$ *satisfy*

$$\int_0^T (\|\boldsymbol{\Sigma}_{XX}(t)\|_2^2 + \|\boldsymbol{\Sigma}_{XY}(t)\|_2^2 + \|\boldsymbol{\Sigma}_{YX}(t)\|_2^2 + \|\boldsymbol{\Sigma}_{YY}(t)\|_2^2)\,\mathrm{d}t < \infty.$$

The norms $\|\cdot\|_p$ *are defined via* $\|\boldsymbol{A}\|_p = \big(\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^p\big)^{1/p}$, *where* $\boldsymbol{A} = [a_{ij}]_{1\le i\le n, 1\le j\le m}$. *Further, denote* $\boldsymbol{\Sigma}_Y(t) = (\boldsymbol{\Sigma}_{YX}(t), \boldsymbol{\Sigma}_{YY}(t))$ *and assume that the matrices* $\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^T(t)$ *are uniformly non-singular,* $0 \le t \le T$. *Then, the filtered process* $\widehat{\underline{X}} = (\widehat{\underline{X}}(t))_{t\ge 0}$, *defined via* $\widehat{\underline{X}}(t) = \mathrm{E}[\underline{X}(t)|\mathcal{F}_t^{\underline{Y}}]$ *and where* $\mathcal{F}_t^{\underline{Y}} = \sigma(\underline{Y}(s); s \le t)$, *satisfies the stochastic differential equation*

$$\mathrm{d}\widehat{\underline{X}}(t) = [\underline{b}_X(t) + \boldsymbol{A}_X(t)\widehat{\underline{Z}}(t)]\,\mathrm{d}t + \boldsymbol{G}(t)[\mathrm{d}\underline{Y}(t) - [\underline{b}_Y(t) + \boldsymbol{A}_Y(t)\widehat{\underline{Z}}(t)]\,\mathrm{d}t],$$

where $\boldsymbol{A}_X(t) = (\boldsymbol{A}_{XX}(t), \boldsymbol{A}_{XY}(t))$, $\boldsymbol{A}_Y(t) = (\boldsymbol{A}_{YX}(t), \boldsymbol{A}_{YY}(t))$, $\widehat{\underline{Z}}(t) = (\widehat{\underline{X}}(t), \underline{Y}(t))$, *and*

$$\boldsymbol{G}(t) = [\boldsymbol{\Sigma}_X(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t) + \boldsymbol{\Gamma}(t)\boldsymbol{A}_{YX}^{\mathsf{T}}(t)](\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t))^{-1}, \qquad \boldsymbol{\Sigma}_X(t) = (\boldsymbol{\Sigma}_{XX}(t), \boldsymbol{\Sigma}_{XY}(t)),$$

and where $\boldsymbol{\Gamma}(t)$ *satisfies the Riccati equation*

$$\mathrm{d}\boldsymbol{\Gamma}(t) = [\boldsymbol{A}_{XX}(t)\boldsymbol{\Gamma}(t) + \boldsymbol{\Gamma}(t)\boldsymbol{A}_{XX}^{\mathsf{T}}(t) + \boldsymbol{\Sigma}_X(t)\boldsymbol{\Sigma}_X^{\mathsf{T}}(t) - \boldsymbol{G}(t)\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)\boldsymbol{G}^{\mathsf{T}}(t)]\,\mathrm{d}t.$$

By using the same notation as in Theorem C.10, we may state the following useful result.

Lemma C.11. *In the setup of Theorem C.10, there exists an* $\mathcal{F}^{\underline{Y}}$*-measurable Wiener process* $\widehat{\underline{W}}$ *and a symmetric non-singular matrix* $\boldsymbol{Q}_Y(t)$ *satisfying*

$$\boldsymbol{Q}_Y(t)\,\mathrm{d}\widehat{\underline{W}}(t) = \mathrm{d}\underline{Y}(t) - [\underline{b}_Y(t) + \boldsymbol{A}_{YX}(t)\widehat{\underline{X}}(t) + \boldsymbol{A}_{YY}(t)\underline{Y}(t)]\,\mathrm{d}t,$$

and $\boldsymbol{Q}_Y(t)\boldsymbol{Q}_Y^{\mathsf{T}}(t) = \boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^T(t)$, *for all* $0 \le t \le T$. *Hence, the filtered process* $\widehat{\underline{X}} = (\widehat{\underline{X}}(t))_{t\ge 0}$ *satisfies the stochastic differential equation*

$$\mathrm{d}\widehat{\underline{X}}(t) = [\underline{b}_X(t) + \boldsymbol{A}_{XX}(t)\widehat{\underline{X}}(t) + \boldsymbol{A}_{XY}(t)\underline{Y}(t)]\,\mathrm{d}t + \boldsymbol{G}(t)\boldsymbol{Q}_Y(t)\,\mathrm{d}\widehat{\underline{W}}(t).$$

Proof. We first note that due to non-singularity $\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)$ is positive definite. Indeed, for any non-zero vector $\underline{x} \in \mathbb{R}^l$ we have $\underline{x}^{\mathsf{T}}\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)\underline{x} = (\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)\underline{x})^{\mathsf{T}}\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)\underline{x} \ge 0$, which implies that $\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)$ is positive semi-definite. However, since $\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)$ is non-singular it has full rank, which implies $\boldsymbol{\Sigma}_Y(t)\boldsymbol{\Sigma}_Y^{\mathsf{T}}(t)\underline{x} \neq \underline{0}$ for any non-zero vector $\underline{x} \in \mathbb{R}^l$, and therefore is positive definite.

Now, since $\mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y^\mathsf{T}(t)$ is real-valued and symmetric, we may write its eigendecomposition as $\mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y^\mathsf{T}(t) = \mathbf{U}(t)\mathbf{D}(t)\mathbf{U}(t)^\mathsf{T}$, where $\mathbf{U}(t)$ is an orthogonal matrix and $\mathbf{D}(t)$ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y(t)^\mathsf{T}$. Since $\mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y^\mathsf{T}(t)$ is positive definite, all its eigenvalues are strictly positive, and thus we may define the diagonal matrix $\mathbf{D}(t)^{1/2}$ whose entries are the square roots of the eigenvalues of $\mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y^\mathsf{T}(t)$. We then define the matrix $\boldsymbol{Q}_Y(t) = \mathbf{U}(t)\mathbf{D}(t)^{1/2}\mathbf{U}(t)^\mathsf{T}$, which clearly is symmetric and non-singular. Moreover, we have

$$\begin{aligned}\boldsymbol{Q}_Y(t)\boldsymbol{Q}_Y^\mathsf{T}(t) &= \mathbf{U}(t)\mathbf{D}(t)^{1/2}\mathbf{U}(t)^\mathsf{T}\mathbf{U}(t)\mathbf{D}(t)^{1/2}\mathbf{U}(t)^\mathsf{T} \\ &= \mathbf{U}(t)\mathbf{D}(t)^{1/2}\mathbf{D}(t)^{1/2}\mathbf{U}(t)^\mathsf{T} = \mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y^\mathsf{T}(t),\end{aligned}$$

which implies that there exists a symmetric non-singular matrix $\boldsymbol{Q}_Y(t)$ satisfying the identity $\boldsymbol{Q}_Y(t)\boldsymbol{Q}_Y^\mathsf{T}(t) = \mathbf{\Sigma}_Y(t)\mathbf{\Sigma}_Y^T(t)$.

So we may define the process $\underline{\widehat{W}}$ by

$$\underline{\widehat{W}}(t) = \int_0^t \boldsymbol{Q}_Y^{-1}(s)\,\mathrm{d}\underline{Y}(s) - \int_0^t \boldsymbol{Q}_Y^{-1}(s)[\underline{b}_Y(s) + \boldsymbol{A}_Y(s)\underline{\widehat{Z}}(s)]\,\mathrm{d}s,$$

where we set $\boldsymbol{A}_Y = (\boldsymbol{A}_{YX}, \boldsymbol{A}_{YY})$ and $\underline{\widehat{Z}} = (\underline{\widehat{X}}, \underline{Y})$. Since $\underline{\widehat{X}}(t) = \mathrm{E}[\underline{X}(t)|\mathcal{F}_t^{\underline{Y}}]$, $t \geq 0$, the process $\underline{\widehat{W}}$ is $\mathcal{F}^{\underline{Y}}$-adapted. We then note that

$$\mathrm{d}\underline{Y}(t) - [\underline{b}_Y(t) + \boldsymbol{A}_Y(t)\underline{\widehat{Z}}(t)]\,\mathrm{d}t = \boldsymbol{A}_{YX}(t)[\underline{X}(t) - \underline{\widehat{X}}(t)]\,\mathrm{d}t + \mathbf{\Sigma}_Y\,\mathrm{d}\underline{W}(t),$$

where we set $\underline{W} = (\underline{W}^X, \underline{W}^Y)$. In particular, we have that

$$\underline{\widehat{W}}(t+h) - \underline{\widehat{W}}(t) = \int_t^{t+h} \boldsymbol{Q}_Y^{-1}(s)\boldsymbol{A}_{YX}(s)[\underline{X}(s) - \underline{\widehat{X}}(s)]\,\mathrm{d}s + \int_t^{t+h} \boldsymbol{Q}_Y^{-1}(s)\mathbf{\Sigma}_Y(s)\,\mathrm{d}\underline{W}(s).$$

So we obtain that $\mathrm{E}[\underline{\widehat{W}}(t+h) - \underline{\widehat{W}}(t)|\mathcal{F}_t^{\underline{Y}}] = 0$, which implies that $\underline{\widehat{W}}$ is a local martingale. On the other hand, we have that

$$\langle \underline{\widehat{W}}, \underline{\widehat{W}}^\mathsf{T}\rangle_t = \langle \boldsymbol{Q}_Y^{-1}\mathbf{\Sigma}_Y\underline{W}, \underline{W}^\mathsf{T}\mathbf{\Sigma}_Y^\mathsf{T}(\boldsymbol{Q}_Y^{-1})^\mathsf{T}\rangle_t = \boldsymbol{Q}_Y^{-1}\mathbf{\Sigma}_Y\langle \underline{W}, \underline{W}^\mathsf{T}\rangle_t\mathbf{\Sigma}_Y^\mathsf{T}(\boldsymbol{Q}_Y^{-1})^\mathsf{T} = t\boldsymbol{I},$$

where the last identity follows from $\boldsymbol{Q}_Y^{-1}\mathbf{\Sigma}_Y\mathbf{\Sigma}_Y^\mathsf{T}(\boldsymbol{Q}_Y^{-1})^\mathsf{T} = \boldsymbol{Q}_Y^{-1}\boldsymbol{Q}_Y\boldsymbol{Q}_Y^\mathsf{T}(\boldsymbol{Q}_Y^\mathsf{T})^{-1} = \boldsymbol{I}$. In particular, we have that $\langle \widehat{W}^k, \widehat{W}^l\rangle_t = \delta_{kl}t$. Hence, $\underline{\widehat{W}}$ is a Wiener process by Lévy's characterization of Brownian motion. ■

In a next step, we are interested in how the matrices $\boldsymbol{Q}^n$ and $\boldsymbol{Q}^{a_i,n}$ of the filters $\widehat{G}^n$ and $\widehat{G}^{a_i,n}$ actually look like. We have the following result.

Lemma C.12. *The matrices $\boldsymbol{Q}^n$ and $\boldsymbol{Q}^{a_i,n}$ are given by*

$$\boldsymbol{Q}^n = \begin{pmatrix} \sigma_D & 0 \\ 0 & (\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2)^{1/2} \end{pmatrix},$$

and

$$\boldsymbol{Q}^{a_i,n} = \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & \frac{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)^{1/2}m_{a_i,n}+\mu_n^2\sigma_G^2+\sigma_{\xi,n}^2}{\lambda_{a_i,n,+}^{1/2}+\lambda_{a_i,n,-}^{1/2}} & \frac{\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2}{\lambda_{a_i,n,+}^{1/2}+\lambda_{a_i,n,-}^{1/2}} \\ 0 & \frac{\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2}{\lambda_{a_i,n,+}^{1/2}+\lambda_{a_i,n,-}^{1/2}} & \frac{(\tilde{\mu}_{a_i,n}^2\sigma_G^2+\tilde{\sigma}_{a_i,n}^2)^{1/2}m_{a_i,n}+\overline{\nu}_{a_i,n}^2\sigma_G^2+m_{a_i,n}^2}{\lambda_{a_i,n,+}^{1/2}+\lambda_{a_i,n,-}^{1/2}} \end{pmatrix},$$

with

$$\lambda_{a_i,n,\pm} = \frac{(\mu_n^2+\overline{\nu}_{a_i,n}^2)\sigma_G^2+\sigma_{\xi,n}^2+m_{a_i,n}^2\pm\sqrt{((\mu_n^2-\overline{\nu}_{a_i,n}^2)\sigma_G^2+\sigma_{\xi,n}^2-m_{a_i,n}^2)^2+4(\mu_n\overline{\nu}_{a_i,n}\sigma_G^2+w(a_i)m_{a_i,n}^2)^2}}{2}.$$

Moreover, they both converge to the "same" matrix $\boldsymbol{Q}$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \boldsymbol{Q}^{a_i,n} = \begin{pmatrix} \boldsymbol{Q} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}, \qquad \lim_{n\to\infty} \boldsymbol{Q}^n = \boldsymbol{Q} =: \begin{pmatrix} \sigma_D & 0 \\ 0 & (\mu^2\sigma_G^2 + \sigma_\xi^2)^{1/2} \end{pmatrix}.$$

Proof. From Lemma C.11 we know that the matrix $\boldsymbol{Q}$ is symmetric and non-singular, satisfying $\boldsymbol{Q}\boldsymbol{Q}^\top = \boldsymbol{\Sigma}_Y\boldsymbol{\Sigma}_Y^\top$. In the case of the public filter, we obtain from the proof of Lemma C.2 that

$$\boldsymbol{Q}^n\boldsymbol{Q}^n = \begin{pmatrix} \sigma_D^2 & 0 \\ 0 & \mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2 \end{pmatrix}.$$

On the other hand, from the proof of Lemma C.1, we know that

$$\boldsymbol{Q}^{a_i,n}\boldsymbol{Q}^{a_i,n} = \begin{pmatrix} \sigma_D^2 & 0 & 0 \\ 0 & \mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2 & \mu_n\overline{\nu}_{a_i,n}\sigma_G^2 + w(a_i)m_{a_i,n}^2 \\ 0 & \mu_n\overline{\nu}_{a_i,n}\sigma_G^2 + w(a_i)m_{a_i,n}^2 & \overline{\nu}_{a_i,n}^2\sigma_G^2 + m_{a_i,n}^2 \end{pmatrix}.$$

To see how the matrices $\boldsymbol{Q}^n$ and $\boldsymbol{Q}^{a_i,n}$ can be defined, let us consider a symmetric positive semi-definite matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ with diagonal entries a and c and the off-diagonal element given by b. The eigenvalues of $\boldsymbol{A}$ are given by

$$\lambda_1 = \frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}, \qquad \lambda_2 = \frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}.$$

We then define

$$\mathbf{U} := \begin{pmatrix} \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{-(\lambda_1-a)}{((\lambda_1-a)^2+b^2)^{1/2}} \\ \frac{\lambda_1-a}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} \end{pmatrix}, \qquad \mathbf{D} := \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix},$$

so that

$$\mathbf{U}\mathbf{U}^\top = \mathbf{U}^\top\mathbf{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{D}\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and therefore $\mathbf{U}\mathbf{D}\mathbf{U}^\top(\mathbf{U}\mathbf{D}\mathbf{U}^\top)^\top = \mathbf{U}\mathbf{D}\mathbf{U}^\top\mathbf{U}\mathbf{D}\mathbf{U}^\top = \mathbf{U}\mathbf{D}\mathbf{D}\mathbf{U}^\top$, with

$$\begin{aligned}
\mathbf{U}\mathbf{D}\mathbf{D}\mathbf{U}^\top &= \begin{pmatrix} \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{-(\lambda_1-a)}{((\lambda_1-a)^2+b^2)^{1/2}} \\ \frac{\lambda_1-a}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{\lambda_1-a}{((\lambda_1-a)^2+b^2)^{1/2}} \\ \frac{-(\lambda_1-a)}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{-(\lambda_1-a)}{((\lambda_1-a)^2+b^2)^{1/2}} \\ \frac{\lambda_1-a}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{b}{((\lambda_1-a)^2+b^2)^{1/2}} \end{pmatrix} \begin{pmatrix} \frac{\lambda_1 b}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{\lambda_1(\lambda_1-a)}{((\lambda_1-a)^2+b^2)^{1/2}} \\ \frac{-\lambda_2(\lambda_1-a)}{((\lambda_1-a)^2+b^2)^{1/2}} & \frac{\lambda_2 b}{((\lambda_1-a)^2+b^2)^{1/2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\lambda_1 b^2+\lambda_2(\lambda_1-a)^2}{(\lambda_1-a)^2+b^2} & \frac{\lambda_1 b(\lambda_1-a)-\lambda_2 b(\lambda_1-a)}{(\lambda_1-a)^2+b^2} \\ \frac{\lambda_1 b(\lambda_1-a)-\lambda_2 b(\lambda_1-a)}{(\lambda_1-a)^2+b^2} & \frac{\lambda_2 b^2+\lambda_1(\lambda_1-a)^2}{(\lambda_1-a)^2+b^2} \end{pmatrix}.
\end{aligned}$$

However, we may notice that

$$\frac{\lambda_1 b(\lambda_1-a) - \lambda_2 b(\lambda_1-a)}{(\lambda_1-a)^2+b^2} = \frac{(\lambda_1-a)(\lambda_1-\lambda_2)}{(\lambda_1-a)^2+b^2} b = \frac{(\lambda_1-a)\sqrt{(a-c)^2+4b^2}}{(\lambda_1-a)^2+b^2} b,$$

which together with

$$\begin{aligned}
(\lambda_1-a)^2+b^2 &= \frac{(c-a)^2+2(c-a)\sqrt{(a-c)^2+4b^2}+(a-c)^2+4b^2}{4}+b^2 \\
&= \frac{(c-a)^2+4b^2+(c-a)\sqrt{(a-c)^2+4b^2}}{2} \\
&= \sqrt{(a-c)^2+4b^2}\,\frac{c-a+\sqrt{(c-a)^2+4b^2}}{2} = (\lambda_1-a)\sqrt{(a-c)^2+4b^2},
\end{aligned}$$

yields

$$\frac{\lambda_1 b(\lambda_1-a)-\lambda_2 b(\lambda_1-a)}{(\lambda_1-a)^2+b^2} = b.$$

On the other hand, we have $\lambda_1+\lambda_2 = a+c$ and $\lambda_1-\lambda_2 = \sqrt{(a-c)^2+4b^2}$, so that

$$\begin{aligned}
\lambda_1 b^2 + \lambda_2(\lambda_1-a)^2 &= \lambda_1 b^2 + \lambda_1(\lambda_1-a)^2 - (\lambda_1-\lambda_2)(\lambda_1-a)^2 \\
&= \lambda_1(b^2+(\lambda_1-a)^2) - \sqrt{(a-c)^2+4b^2}(\lambda_1-a)^2 \\
&= \lambda_1(b^2+(\lambda_1-a)^2) - (\lambda_1-a)(b^2+(\lambda_1-a)^2) = a(b^2+(\lambda_1-a)^2),
\end{aligned}$$

as well as

$$\begin{aligned}\lambda_2 b^2 + \lambda_1(\lambda_1 - a)^2 &= \lambda_2 b^2 + \lambda_2(\lambda_1 - a)^2 + (\lambda_1 - \lambda_2)(\lambda_1 - a)^2 \\ &= \lambda_2(b^2 + (\lambda_1 - a)^2) - (\lambda_1 - a)\sqrt{(a-c)^2 + 4b^2}(\lambda_2 - c) \\ &= \lambda_2(b^2 + (\lambda_1 - a)^2) - (\lambda_2 - c)(b^2 + (\lambda_1 - a)^2) = c(b^2 + (\lambda_1 - a)^2).\end{aligned}$$

Therefore, we eventually obtain $\mathbf{UDDU}^\mathsf{T} = \boldsymbol{A}$. In particular, we have

$$\begin{aligned}&\mathbf{UDU}^\mathsf{T} \\ &= \begin{pmatrix} \frac{\lambda_1^{1/2} b^2 + \lambda_2^{1/2}(\lambda_1 - a)^2}{(\lambda_1-a)^2+b^2} & \frac{\lambda_1^{1/2} b(\lambda_1 - a) - \lambda_2^{1/2} b(\lambda_1 - a)}{(\lambda_1-a)^2+b^2} \\ \frac{\lambda_1^{1/2} b(\lambda_1 - a) - \lambda_2^{1/2} b(\lambda_1 - a)}{(\lambda_1-a)^2+b^2} & \frac{\lambda_2^{1/2} b^2 + \lambda_1^{1/2}(\lambda_1 - a)^2}{(\lambda_1-a)^2+b^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\lambda_1^{1/2} b^2 + \lambda_1^{1/2}(\lambda_1 - a)^2}{(\lambda_1-a)^2+b^2} - \frac{(\lambda_1-\lambda_2)(\lambda_1-a)}{(\lambda_1-a)^2+b^2} \cdot \frac{\lambda_1 - a}{\lambda_1^{1/2}+\lambda_2^{1/2}} & \frac{(\lambda_1-a)(\lambda_1-\lambda_2)}{(\lambda_1-a)^2+b^2} \cdot \frac{b}{\lambda_1^{1/2}+\lambda_2^{1/2}} \\ \frac{(\lambda_1-a)(\lambda_1-\lambda_2)}{(\lambda_1-a)^2+b^2} \cdot \frac{b}{\lambda_1^{1/2}+\lambda_2^{1/2}} & \frac{\lambda_2^{1/2} b^2 + \lambda_2^{1/2}(\lambda_1 - a)^2}{(\lambda_1-a)^2+b^2} - \frac{(\lambda_2-\lambda_1)(\lambda_1-a)}{(\lambda_1-a)^2+b^2} \cdot \frac{\lambda_1 - a}{\lambda_1^{1/2}+\lambda_2^{1/2}} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1^{1/2} - \frac{\lambda_1 - a}{\lambda_1^{1/2}+\lambda_2^{1/2}} & \frac{b}{\lambda_1^{1/2}+\lambda_2^{1/2}} \\ \frac{b}{\lambda_1^{1/2}+\lambda_2^{1/2}} & \lambda_2^{1/2} - \frac{\lambda_2 - c}{\lambda_1^{1/2}+\lambda_2^{1/2}} \end{pmatrix} = \begin{pmatrix} \frac{\det(\boldsymbol{A})^{1/2}+a}{\lambda_1^{1/2}+\lambda_2^{1/2}} & \frac{b}{\lambda_1^{1/2}+\lambda_2^{1/2}} \\ \frac{b}{\lambda_1^{1/2}+\lambda_2^{1/2}} & \frac{\det(\boldsymbol{A})^{1/2}+c}{\lambda_1^{1/2}+\lambda_2^{1/2}} \end{pmatrix}.\end{aligned}$$

Hence, we may use such a procedure, which in essence uses the eigenvalue decomposition of a matrix, to obtain $\boldsymbol{Q}^{a_i,n}$ of the form

$$\boldsymbol{Q}^{a_i,n} = \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & \frac{(a_n c_n - b_n^2)^{1/2} + a_n}{\lambda_{1,n}^{1/2}+\lambda_{2,n}^{1/2}} & \frac{b_n}{\lambda_{1,n}^{1/2}+\lambda_{2,n}^{1/2}} \\ 0 & \frac{b_n}{\lambda_{1,n}^{1/2}+\lambda_{2,n}^{1/2}} & \frac{(a_n c_n - b_n^2)^{1/2} + c_n}{\lambda_{1n}^{1/2}+\lambda_{2,n}^{1/2}} \end{pmatrix},$$

with $a_n = \mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2$, $b_n = \mu_n \overline{\nu}_{a_i,n}\sigma_G^2 + w(a_i)m_{a_i,n}^2$, and $c_n = \overline{\nu}_{a_i,n}^2\sigma_G^2 + m_{a_i,n}^2$, and the eigenvalues $\lambda_{1,n}$ and $\lambda_{2,n}$ as defined above. In the limit as $n \to \infty$, we then observe that $a_n \to \mu^2\sigma_G^2 + \sigma_\xi^2$, $b_n \to 0$ and $c_n \to 0$, so that $\lambda_{1,n} \to \mu^2\sigma_G^2 + \sigma_\xi^2$ and $\lambda_{2,n} \to 0$, and therefore

$$\lim_{n\to\infty} \boldsymbol{Q}^{a_i,n} = \begin{pmatrix} \sigma_D & 0 & 0 \\ 0 & (\mu^2\sigma_G^2 + \sigma_\xi^2)^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the matrix $\boldsymbol{Q}$ is the limit of $\boldsymbol{Q}^n$ as $n \to \infty$, which comes from the public filter, i.e.,

$$\boldsymbol{Q} := \lim_{n\to\infty} \boldsymbol{Q}^n = \lim_{n\to\infty} \begin{pmatrix} \sigma_D & 0 \\ 0 & (\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2)^{1/2} \end{pmatrix}.$$

This concluses the proof. ■

D. Measure changes

Our goal here is to establish the equivalent probability measure $P^{a_i,n} \approx P$ of investor a_i in the discrete market with n investors. We do this by first constructing the measure $P^{a_i,n}$ on a finite time interval $[0,T]$ and then extend it to the positive half-line.

D.1. The finite time horizon

Here we focus on finite time horizons $[0,T]$ for some arbitrary $T \geq 0$.

Proposition D.1. *Fix $T \geq 0$ and consider the vector functions*

$$\underline{b}(a_i,\cdot)\colon \mathbb{R} \to \mathbb{R}^{3\times 1},\ t \mapsto \underline{b}(a_i,t), \qquad \underline{\theta}(a_i,\cdot)\colon \mathbb{R} \to \mathbb{R}^{3\times 1},\ t \mapsto \underline{\theta}(a_i,t),$$

for some $a_i \in [0,1)$. Assume that

$$\int_0^T \underline{\theta}^\mathsf{T}(a_i,t)\underline{\theta}(a_i,t)\,\mathrm{d}t < \infty, \quad \sup_{0\leq t\leq T} (\underline{\theta}^\mathsf{T}(a_i,t)\underline{b}(a_i,t))^2 < \infty, \quad \sup_{0\leq t\leq T} \underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t) < \infty.$$

Then the probability measure $\mathrm{d}P^{a_i,n} = Z_T^{a_i,n}\,\mathrm{d}P$, where

$$Z_T^{a_i,n} = \mathrm{e}^{\int_0^T (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))^\mathsf{T}\,\mathrm{d}\widehat{\underline{W}}_s^{a_i,n} - \frac{1}{2}\int_0^T (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))^\mathsf{T}(\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))\,\mathrm{d}s},$$

is equivalent to P, and the process $\widetilde{\underline{W}}^{a_i,n} = (\widetilde{\underline{W}}_t^{a_i,n})_{0\leq t\leq T}$, defined as

$$\widetilde{\underline{W}}_t^{a_i,n} = \widehat{\underline{W}}_t^{a_i,n} - \int_0^t (\underline{\theta}(a_i,s) + \Lambda_s^{a_i,n}\underline{b}(a_i,s))\,\mathrm{d}s,$$

is an $\mathbb{R}^3$-valued Brownian motion under $P^{a_i,n}$.

Proof. From Corollary C.3 together with Lemma C.11 we know that the stochastic process $\Lambda^{a_i,n} = (\Lambda_t^{a_i,n})_{t\geq 0}$ satisfies the stochastic differential equation

$$\mathrm{d}\Lambda_t^{a_i,n} = -\alpha_\Lambda^n(t)\Lambda_t^{a_i,n}\,\mathrm{d}t + \left(\mathbf{\Gamma}_t^{a_i,n} - \begin{pmatrix}\mathbf{\Gamma}_t^n & 0\end{pmatrix}\right)\boldsymbol{Q}^{a_i,n}\,\mathrm{d}\widehat{\underline{W}}^{a_i,n}(t),$$

where $\alpha_\Lambda^n(t)$, $\mathbf{\Gamma}_t^{a_i,n}$ and $\mathbf{\Gamma}_t^n$ are defined as in Lemma C.1, Lemma C.2 and Corollary C.3, respectively. From Itô's lemma we thus obtain

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{\int_0^t \alpha_\Lambda^n(s)\,\mathrm{d}s}\,\Lambda_t^{a_i,n}\right) &= \alpha_\Lambda^n(t)\,\mathrm{e}^{\int_0^t \alpha_\Lambda^n(s)\,\mathrm{d}s}\,\Lambda_t^{a_i,n}\,\mathrm{d}t + \mathrm{e}^{\int_0^t \alpha_\Lambda^n(s)\,\mathrm{d}s}\,\mathrm{d}\Lambda_t^{a_i,n} \\ &= \mathrm{e}^{\int_0^t \alpha_\Lambda(s)\,\mathrm{d}s}\left(\mathbf{\Gamma}_t^{a_i,n} - \begin{pmatrix}\mathbf{\Gamma}_t^n & 0\end{pmatrix}\right)\boldsymbol{Q}^{a_i,n}\,\mathrm{d}\widehat{\underline{W}}_t^{a_i,n},\end{aligned}$$

which implies

$$\Lambda_t^{a_i,n} = \Lambda_0^{a_i,n}\,\mathrm{e}^{-\int_0^t \alpha_\Lambda^n(s)\,\mathrm{d}s} + \int_0^t \mathrm{e}^{-\int_s^t \alpha_\Lambda^n(u)\,\mathrm{d}u} \left(\boldsymbol{\Gamma}_s^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}_s^n & 0\end{pmatrix}\right) \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n}.$$

Since the private signal at $t=0$ is equal to zero because $\zeta_0^{a_i,n} = \Delta_1 W^\xi_{(a_i,0)}$, the above equation reduces to

$$\Lambda_t^{a_i,n} = \int_0^t \mathrm{e}^{-\int_s^t \alpha_\Lambda^n(u)\,\mathrm{d}u} \left(\boldsymbol{\Gamma}_s^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}_s^n & 0\end{pmatrix}\right) \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n}.$$

As a result, we obtain

$$\mathrm{E}[(\Lambda_t^{a_i,n})^2] = \int_0^t \mathrm{e}^{-2\int_s^t \alpha_\Lambda^n(u)\,\mathrm{d}u}\,\eta_{\Lambda\Lambda}^{a_i,n}(s)\,\mathrm{d}s,$$

where we denote

$$\eta_{\Lambda\Lambda}^{a_i,n}(t) = \left(\boldsymbol{\Gamma}_t^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}_t^n & 0\end{pmatrix}\right) \boldsymbol{Q}^{a_i,n}\boldsymbol{Q}^{a_i,n} \left(\boldsymbol{\Gamma}_t^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}_t^n & 0\end{pmatrix}\right)^{\mathsf{T}}.$$

The function $\eta_{\Lambda\Lambda}^{a_i,n}$ only depends on t via $\boldsymbol{\Gamma}_t^{a_i,n}$ and $\boldsymbol{\Gamma}_t^n$, which themselves only depend on t via the mean squared errors $\gamma_t^{a_i,n}$ and γ_t^n, respectively. Since $\gamma_t^{a_i,n}$ and γ_t^n are increasing in t with $\gamma^{a_i,n} = \lim_{t\to\infty}\gamma_t^{a_i,n}$ and $\gamma^n = \lim_{t\to\infty}\gamma_t^n$ finite, we thus obtain that $\sup_{0\le t\le T}\eta_{\Lambda\Lambda}^{a_i,n}(t) < \infty$ for each $T \ge 0$. On the other hand, we know from Corollary C.3 that $\alpha_\Lambda^n(t) > 0$ is increasing in t with $\lim_{t\to\infty}\alpha_\Lambda^n(t)$ finite. Therefore, we find that

$$\begin{aligned}
\mathrm{E}[(\Lambda_t^{a_i,n})^2] &\le \sup_{0\le s\le t}\eta_{\Lambda\Lambda}^{a_i,n}(s)\int_0^t \mathrm{e}^{-2\int_s^t \alpha_\Lambda^n(u)\,\mathrm{d}u}\,\mathrm{d}s \le \sup_{0\le s\le t}\eta_{\Lambda\Lambda}^{a_i,n}(s)\int_0^t \mathrm{e}^{-2(t-s)\inf_{0\le u\le T}\alpha_\Lambda^n(u)}\,\mathrm{d}s \\
&= \sup_{0\le s\le t}\eta_{\Lambda\Lambda}^{a_i,n}(s)\frac{1-\mathrm{e}^{-2t\inf_{0\le u\le T}\alpha_\Lambda^n(u)}}{2\inf_{0\le u\le T}\alpha_\Lambda^n(u)} \le \sup_{0\le s\le t}\frac{\eta_{\Lambda\Lambda}^{a_i,n}(s)}{2\alpha_\Lambda^n(0)} < \infty,
\end{aligned}$$

which eventually yields

$$\mathrm{E}[(\Lambda_t^{a_i,n})^2]\underline{b}^{\mathsf{T}}(a_i,t)\underline{b}(a_i,t) \le \sup_{0\le s\le T}\frac{\eta_{\Lambda\Lambda}^{a_i,n}(s)}{2\alpha_\Lambda^n(0)}\sup_{0\le s\le T}\underline{b}^{\mathsf{T}}(a_i,s)\underline{b}(a_i,s) \qquad \forall\, 0\le t\le T.$$

Now, define the processes $L^{a_i,n} = (L_t^{a_i,n})_{0\le t\le T}$ and $Z^{a_i,n} = (Z_t^{a_i,n})_{0\le t\le T}$ via

$$L_t^{a_i,n} \coloneqq \int_0^t (\underline{\theta}(a_i,s) + \Lambda_s^{a_i,n}\underline{b}(a_i,s))^{\mathsf{T}}\,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n}, \qquad L_0^{a_i,n} = 0,$$

and $Z_t^{a_i,n} := \mathcal{E}(L^{a_i,n})_t$ the stochastic exponential of $L^{a_i,n}$, that is

$$
\begin{aligned}
Z_t^{a_i,n} &= \exp\left(L_t^{a_i,n} - \frac{1}{2}\langle L^{a_i,n}, L^{a_i,n}\rangle_t\right) \\
&= \mathrm{e}^{\int_0^t (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))^\mathsf{T}\,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n} - \frac{1}{2}\int_0^t (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))^\mathsf{T}(\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))\,\mathrm{d}s}.
\end{aligned}
$$

By assumption, we have

$$
\int_0^T \underline{\theta}^\mathsf{T}(a_i,s)\underline{\theta}(a_i,s)\,\mathrm{d}s < \infty, \qquad \int_0^T \underline{b}^\mathsf{T}(a_i,s)\underline{b}(a_i,s))\,\mathrm{d}s < \infty,
$$

and therefore

$$
\begin{aligned}
&\mathrm{E}[\langle L^{a_i,n}, L^{a_i,n}\rangle_T] \\
&\quad = \int_0^T (\underline{\theta}^\mathsf{T}(a_i,s)\underline{\theta}(a_i,s) + 2\underline{\theta}^\mathsf{T}(a_i,s)\underline{b}(a_i,s)\,\mathrm{E}[\Lambda_s^{a_i,n}] + \mathrm{E}[(\Lambda_s^{a_i,n})^2]\underline{b}^\mathsf{T}(a_i,s)\underline{b}(a_i,s))\,\mathrm{d}s \\
&\quad \le \int_0^T \underline{\theta}^\mathsf{T}(a_i,s)\underline{\theta}(a_i,s)\,\mathrm{d}s + \sup_{0\le s\le T}\frac{\eta_{\Lambda\Lambda}^{a_i,n}(s)}{2\alpha_\Lambda(0)}\int_0^T \underline{b}^\mathsf{T}(a_i,s)\underline{b}(a_i,s))\,\mathrm{d}s < \infty.
\end{aligned}
$$

Hence, by Corollary IV.1.25 of Revuz and Yor (1999), the process $L^{a_i,n}$ is an $L^2(P)$-bounded martingale. In fact, we obtain by Markov's inequality

$$
\begin{aligned}
P[\langle L^{a_i,n}, L^{a_i,n}\rangle_T \ge k] &\le \frac{\mathrm{E}[\langle L^{a_i,n}, L^{a_i,n}\rangle_T]}{k} \\
&\le \frac{1}{k}\left(\int_0^T \underline{\theta}^\mathsf{T}(a_i,s)\underline{\theta}(a_i,s)\,\mathrm{d}s + \sup_{0\le s\le T}\frac{\eta_{\Lambda\Lambda}^{a_i,n}(s)}{2\alpha_\Lambda^n(0)}\int_0^T \underline{b}^\mathsf{T}(a_i,s)\underline{b}(a_i,s))\,\mathrm{d}s\right),
\end{aligned}
$$

which implies $P[\langle L^{a_i,n}, L^{a_i,n}\rangle_T < \infty] = 1$ and therefore $Z_T^{a_i,n} > 0$ P-a.s. On the other hand, we note that

$$
\mathrm{E}\left[\exp\left(\frac{1}{2}\langle L^{a_i,n}, L^{a_i,n}\rangle_T\right)\right] < \infty, \tag{D.1}
$$

and therefore $Z^{a_i,n}$ is a P-martingale with $\mathrm{E}[Z_T^{a_i,n}] = 1$ (see Theorem 6.1 in Liptser and Shiryaev (2001)). The strict inequality (D.1) can be proven as follows. We first note that $\Lambda_t^{a_i,n} \sim \mathcal{N}(m_t^{a_i,n}, v_t^{a_i,n})$, where

$$
m_t^{a_i,n} := \mathrm{E}[\Lambda_0^{a_i,n}]\,\mathrm{e}^{-\int_0^t \alpha_\Lambda^n(s)\,\mathrm{d}s}, \quad v_t^{a_i,n} := \mathrm{Var}[\Lambda_0^{a_i,n}]\,\mathrm{e}^{-2\int_0^t \alpha_\Lambda^n(s)\,\mathrm{d}s} + \int_0^t \mathrm{e}^{-2\int_s^t \alpha_\Lambda^n(u)\,\mathrm{d}u}\,\eta_{\Lambda\Lambda}^{a_i,n}(s)\,\mathrm{d}s.
$$

Hence, for any $\alpha_t^{a_i,n} \in \mathbb{R}$ and any $\beta_t^{a_i,n} > 0$ satisfying $\beta_t^{a_i,n} v_t^{a_i,n} < \frac{1}{2}$ we obtain

$$
\begin{aligned}
&\mathrm{E}[\mathrm{e}^{\alpha_t^{a_i,n}\Lambda_t^{a_i,n}+\beta_t^{a_i,n}(\Lambda_t^{a_i,n})^2}] \\
&\quad= \mathrm{E}\left[\exp\left(\alpha_t^{a_i,n}(v_t^{a_i,n})^{1/2}\frac{\Lambda_t^{a_i,n}}{(v_t^{a_i,n})^{1/2}} + \beta_t^{a_i,n}v_t^{a_i,n}\frac{(\Lambda_t^{a_i,n})^2}{v_t^{a_i,n}}\right)\right] \\
&\quad= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\alpha_t^{a_i,n}(v_t^{a_i,n})^{1/2}x+\beta_t^{a_i,n}v_t^{a_i,n}x^2}\,\mathrm{e}^{-\frac{1}{2}(x-m_t^{a_i,n}/(v_t^{a_i,n})^{1/2})^2}\,\mathrm{d}x \\
&\quad= \frac{\mathrm{e}^{-\frac{1}{2}(m_t^{a_i,n})^2/(v_t^{a_i,n})}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{(\alpha_t^{a_i,n}(v_t^{a_i,n})^{1/2}+m_t^{a_i,n}/(v_t^{a_i,n})^{1/2})x-\frac{1}{2}(1-2\beta_t^{a_i,n}v_t^{a_i,n})x^2}\,\mathrm{d}x \\
&\quad= \frac{\mathrm{e}^{-\frac{1}{2}(m_t^{a_i,n})^2/(v_t^{a_i,n})}}{(1-2\beta_t^{a_i,n}v_t^{a_i,n})^{1/2}}\,\mathrm{e}^{\frac{(\alpha_t^{a_i,n}(v_t^{a_i,n})^{1/2}+m_t^{a_i,n}/(v_t^{a_i,n})^{1/2})^2}{2(1-2\beta_t^{a_i,n}v_t^{a_i,n})}}.
\end{aligned}
$$

However, as discussed before, it holds that $\Lambda_0^{a_i,n} = 0$, so that the above equation reduces to

$$
\mathrm{E}[\mathrm{e}^{\alpha_t^{a_i,n}\Lambda_t^{a_i,n}+\beta_t^{a_i,n}(\Lambda_t^{a_i,n})^2}] = \frac{\mathrm{e}^{\frac{(\alpha_t^{a_i,n})^2 v_t^{a_i,n}}{2(1-2\beta_t^{a_i,n}v_t^{a_i,n})}}}{(1-2\beta_t^{a_i,n}v_t^{a_i,n})^{1/2}} = \frac{\mathrm{e}^{\frac{(\alpha_t^{a_i,n})^2\,\mathrm{E}[(\Lambda_t^{a_i,n})^2]}{2(1-2\beta_t^{a_i,n}\,\mathrm{E}[(\Lambda_t^{a_i,n})^2])}}}{(1-2\beta_t^{a_i,n}\,\mathrm{E}[(\Lambda_t^{a_i,n})^2])^{1/2}}.
$$

By using Jensen's inequality and Fubini's theorem, we thus obtain

$$
\begin{aligned}
\mathrm{E}\left[\mathrm{e}^{\frac{1}{2}\langle L^{a_i,n},L^{a_i,n}\rangle_T}\right] &= \mathrm{E}\left[\mathrm{e}^{\frac{1}{2}\int_0^T(\underline{\theta}^\mathsf{T}(a_i,t)\underline{\theta}(a_i,t)+2\underline{\theta}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)\Lambda_t^{a_i,n}+(\Lambda_t^{a_i,n})^2\underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t))\,\mathrm{d}t}\right] \\
&= \mathrm{e}^{\frac{1}{2}\int_0^T\underline{\theta}^\mathsf{T}(a_i,t)\underline{\theta}(a_i,t)\,\mathrm{d}t}\,\mathrm{E}\left[\mathrm{e}^{\frac{1}{T}\int_0^T T(\underline{\theta}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)\Lambda_t^{a_i,n}+\frac{1}{2}(\Lambda_t^{a_i,n})^2\underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t))\,\mathrm{d}t}\right] \\
&\le \frac{\mathrm{e}^{\frac{1}{2}\int_0^T\underline{\theta}^\mathsf{T}(a_i,t)\underline{\theta}(a_i,t)\,\mathrm{d}t}}{T}\int_0^T\mathrm{E}\left[\mathrm{e}^{T\underline{\theta}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)\Lambda_t^{a_i,n}+\frac{T}{2}(\Lambda_t^{a_i,n})^2\underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)}\right]\mathrm{d}t \\
&\le \frac{\mathrm{e}^{\frac{1}{2}\int_0^T\underline{\theta}^\mathsf{T}(a_i,t)\underline{\theta}(a_i,t)\,\mathrm{d}t+\frac{T^2\sup_{0\le t\le T}(\underline{\theta}^\mathsf{T}(a_i,t)\underline{b}(a_i,t))^2\,\mathrm{E}[(\Lambda_t^{a_i,n})^2]}{2(1-T\sup_{0\le t\le T}\underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)\,\mathrm{E}[(\Lambda_t^{a_i,n})^2])}}}{(1-T\sup_{0\le t\le T}\underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)\,\mathrm{E}[(\Lambda_t^{a_i,n})^2])^{1/2}} < \infty
\end{aligned}
$$

for all

$$
T < \frac{1}{\sup_{0\le t\le T}\underline{b}^\mathsf{T}(a_i,t)\underline{b}(a_i,t)\,\mathrm{E}[(\Lambda_t^{a_i,n})^2]} =: \delta(T).
$$

Hence, for any $T < \delta(T)$ it holds that $\mathrm{E}[Z_T^{a_i,n}] = \mathrm{E}[\mathcal{E}(L^{a_i,n})_T] = 1$ by Theorem 6.1 of Liptser and Shiryaev (2001) (Novikov's condition). On the other hand, if $T \ge \delta(T)$, we may write

$$
Z_T^{a_i,n} = \mathcal{E}(L^{a_i,n})_T = \prod_{j=0}^{m-1}\mathcal{E}(L^{a_i,n})_{t_j}^{t_{j+1}}, \qquad 0 = t_0 < t_1 < \cdots < t_m = T,
$$

where $\mathcal{E}(L^{a_i,n})_{t_j}^{t_{j+1}} := \mathcal{E}(L_{t_{j+1}}^{a_i,n} - L_{t_j}^{a_i,n})$, and $\max_{0\le j\le m-1}(t_{j+1}-t_j) < \delta(T)$. Then, similarly as before, we find $\mathrm{E}[\mathcal{E}(L^{a_i,n})_{t_j}^{t_{j+1}}] = 1$, and $\mathrm{E}[\mathcal{E}(L^{a_i,n})_{t_j}^{t_{j+1}}|\mathcal{F}_{t_j}^{a_i,n}] = \mathrm{E}[\mathcal{E}(L^{a_i,n})_{t_j}^{t_{j+1}}] = 1$ which follows by independence, and which eventually implies

$$\mathrm{E}[Z_T^{a_i,n}] = \mathrm{E}[\mathrm{E}[Z_T^{a_i,n}|\mathcal{F}_{t_{m-1}}^{a_i,n}]] = \mathrm{E}[Z_{t_{m-1}}^{a_i,n}] = \cdots = \mathrm{E}[Z_{t_1}^{a_i,n}] = 1.$$

As a result, the multivariate version of Girsanov's theorem (see, e.g., Theorem 6.4 of Liptser and Shiryaev (2001)) ensures that $P^{a_i,n}$ defined via $\frac{\mathrm{d}P^{a_i,n}}{\mathrm{d}P} =: Z_T^{a_i,n}$ is a probability measure that is equivalent to P, i.e., $P^{a_i,n} \approx P$, and that the process

$$\underline{\widetilde{W}}_t^{a_i,n} = \underline{\widehat{W}}_t^{a_i,n} - \int_0^t (\underline{\theta}(a_i, s) + \Lambda_s^{a_i,n}\underline{b}(a_i s)\,\mathrm{d}s, \qquad 0 \le t \le T,$$

is a Brownian motion under the probability measure $P^{a_i,n}$. This concludes the proof. ■

D.2. The infinite time horizon

The construction of $P^{a_i,n}$ for a finite time horizon T would also work for an infinite time horizon, if the P-martingale $Z^{a_i,n}$ was uniformly integrable. Unfortunately, this is not the case here, and so we proceed differently. In particular, we use a similar approach as in Section 1.7 of Karatzas and Shreve (1998). For this purpose, we first recall the definition $\mathcal{G}_t^{a_i,n} = \sigma(D_s, \Xi_s^n, \zeta_s^{a_i,n}; 0 \le s \le t)$ and that $\underline{\widehat{W}}_t^{a_i,n}$ is $\mathcal{G}_t^{a_i,n}$-measurable, which implies $\sigma(\underline{\widehat{W}}_s^{a_i,n} \colon 0 \le s \le t) \subseteq \mathcal{G}_t^{a_i,n}$. On the other hand, we know that the process $D = (D_t)_{t\ge 0}$ is a strong solution to the stochastic differential equation

$$\mathrm{d}D_t = \alpha_D(\widehat{G}_t^{a_i,n} - D_t)\,\mathrm{d}t + \begin{pmatrix}1 & 0 & 0\end{pmatrix} \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_t^{a_i,n},$$

which implies that D_t is $\sigma(\underline{\widehat{W}}_s^{a_i,n} \colon 0 \le s \le t)$-measurable. Furthermore, we know that the process $\Xi^n = (\Xi_t^n)_{t\ge 0}$ is a strong solution to the stochastic differential equation

$$\mathrm{d}\Xi_t^n = \mu_n \alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t + \begin{pmatrix}0 & 1 & 0\end{pmatrix} \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_t^{a_i,n},$$

which implies that also Ξ_t^n is $\sigma(\underline{\widehat{W}}_s^{a_i,n} \colon 0 \le s \le t)$-measurable. Finally, the discrete-economy signal process $\zeta^{a_i,n} = (\zeta_t^{a_i,n})_{t\ge 0}$ is a strong solution to the stochastic differential equation

$$\mathrm{d}\zeta_t^{a_i,n} = \overline{\nu}_{a_i,n} \alpha_G(g - \widehat{G}_t^{a_i,n})\,\mathrm{d}t + \begin{pmatrix}0 & 0 & 1\end{pmatrix} \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_t^{a_i,n},$$

so that $\zeta_t^{a_i,n}$ is also $\sigma(\underline{\widehat{W}}^{a_i,n}(s) \colon 0 \le s \le t)$-measurable, and therefore we eventually find that $\mathcal{G}_t^{a_i,n} = \sigma(\underline{\widehat{W}}^{a_i,n}(s) \colon 0 \le s \le t)$ for each $t \ge 0$.

To proceed, our goal is to work with the Wiener measure on the space of continuous, $\mathbb{R}^3$-valued functions $C([0,\infty), \mathbb{R}^3)$. In particular, we consider the Brownian motion $\underline{\widehat{W}}^{a_i,n}$ as a mapping from Ω to the function space $C([0,\infty),\mathbb{R}^3)$, i.e.,

$$\widehat{\underline{W}}^{a_i,n}\colon \Omega \to C([0,\infty),\mathbb{R}^3),\ \omega \mapsto (\omega(t))_{t\geq 0}.$$

Then, we may consider $\widehat{\underline{W}}_t^{a_i,n}$ as the coordinate map

$$\widehat{\underline{W}}_t^{a_i,n}\colon C([0,\infty),\mathbb{R}^3) \to \mathbb{R}^3,\ (\omega(t))_{t\geq 0} \mapsto \omega(t).$$

We then define the σ-field $\mathcal{G}_\infty^{a_i,n}$ by

$$\mathcal{G}_\infty^{a_i,n} := \sigma\Bigg(\bigcup_{0\leq t<\infty} \mathcal{G}_t^{a_i,n}\Bigg) = \sigma\Bigg(\bigcup_{0\leq t<\infty} \sigma(\widehat{\underline{W}}_s^{a_i,n}; s\leq t)\Bigg),$$

so that $(C([0,\infty),\mathbb{R}^3),\mathcal{G}_\infty^{a_i,n})$ becomes a measurable space. In order to get a probability space we equip $(C([0,\infty),\mathbb{R}^3),\mathcal{G}_\infty^{a_i,n})$ with the Wiener measure $\lambda^{a_i,n} := P\circ(\widehat{\underline{W}}^{a_i,n})^{-1}$. Moreover, to work with a complete probability space, we denote the set of $\lambda^{a_i,n}$-nullsets by $\mathcal{N}^{a_i,n}$, i.e.,

$$\mathcal{N}^{a_i,n} := \{N \subseteq C([0,\infty),\mathbb{R}^3)\colon \exists A \in \mathcal{G}_\infty^{a_i,n}, \text{ s.t. } N\subseteq A \text{ and } \lambda^{a_i,n}(A)=0\},$$

and we define $\overline{\mathcal{G}}_\infty^{a_i,n} := \sigma(\mathcal{G}_\infty^{a_i,n}\cup\mathcal{N}^{a_i,n})$. Hence, $(C([0,\infty),\mathbb{R}^3),\overline{\mathcal{G}}_\infty^{a_i,n},\lambda^{a_i,n})$ is a complete probability space. Similarly, we define for each $T\geq 0$ the subspaces $(C([0,\infty),\mathbb{R}^3),\overline{\mathcal{G}}_T^{a_i,n},\lambda_T^{a_i,n})$, where $\lambda_T^{a_i,n}$ is the restriction of $\lambda^{a_i,n}$ to $\overline{\mathcal{G}}_T^{a_i,n}$, and where $\overline{\mathcal{G}}_T^{a_i,n}$ is the $\lambda^{a_i,n}$-completion of $\mathcal{G}_T^{a_i,n}$. In other words, we define the restriction of $\mathcal{N}^{a_i,n}$ to $\mathcal{G}_T^{a_i,n}$-nullsets by

$$\mathcal{N}_T^{a_i,n} := \{N \subseteq C([0,\infty),\mathbb{R}^3)\colon \exists A \in \mathcal{G}_T^{a_i,n}, \text{ s.t. } N\subseteq A \text{ and } \lambda^{a_i,n}(A)=0\},$$

so that $\overline{\mathcal{G}}_T^{a_i,n} = \sigma(\mathcal{G}_T^{a_i,n}\cup\mathcal{N}_T^{a_i,n})$. Further, the augmented filtration $\overline{\mathbb{G}}^{a_i,n,T} = (\overline{\mathcal{G}}_t^{a_i,n,T})_{0\leq t\leq T}$, is defined as $\overline{\mathcal{G}}_t^{a_i,n,T} := \sigma(\mathcal{G}_t^{a_i,n}\cup\mathcal{N}_T^{a_i,n})$.

Now, we consider the processes $Z^{a_i,n} = (Z_t^{a_i,n})_{t\geq 0}$ and $\widetilde{\underline{W}}^{a_i,n} = (\widetilde{\underline{W}}_t^{a_i,n})_{t\geq 0}$, defined via

$$Z_t^{a_i,n} = \mathrm{e}^{\int_0^t (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))^\mathsf{T}\,\mathrm{d}\widehat{\underline{W}}_s^{a_i,n} - \frac{1}{2}\int_0^t (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))^\mathsf{T}(\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))\,\mathrm{d}s},$$

and

$$\widetilde{\underline{W}}_t^{a_i,n} = \widehat{\underline{W}}_t^{a_i,n} - \int_0^t (\underline{\theta}(a_i,s)+\Lambda_s^{a_i,n}\underline{b}(a_i,s))\,\mathrm{d}s.$$

By Proposition D.1 we know that the probability measure $P_T^{a_i,n}$ defined via $\mathrm{d}P_T^{a_i,n} = Z_T^{a_i,n}\,\mathrm{d}P$ is equivalent to P, and that the restricted process $\widetilde{\underline{W}}^{a_i,n,T} = (\widetilde{\underline{W}}_t^{a_i,n})_{0\leq t\leq T}$ is a Brownian motion under $P_T^{a_i,n}$. In fact, for $0\leq t\leq T$, the probability measure $\kappa_T^{a_i,n} := P_T^{a_i,n}\circ(\widetilde{\underline{W}}^{a_i,n,T})^{-1}$

is equivalent to $\lambda^{a_i,n}$ on $\overline{\mathcal{G}}_t^{a_i,n,T}$ because for any $A \in \overline{\mathcal{G}}_t^{a_i,n,T}$ it holds that

$$\begin{aligned}
\kappa_T^{a_i,n}(A) &= P_T^{a_i,n}[\{\omega \in \Omega\colon \underline{\widetilde{W}}^{a_i,n,T}(\omega) \in A\}] = P_T^{a_i,n}[(\underline{\widetilde{W}}^{a_i,n,T})^{-1}(A)] \\
&= \int_{(\underline{\widetilde{W}}^{a_i,n,T})^{-1}(A)} \mathrm{d}P_T^{a_i,n} = \int_\Omega (\mathbb{1}_A \circ \underline{\widetilde{W}}^{a_i,n,T})\,\mathrm{d}P_T^{a_i,n} = \int_\Omega (\mathbb{1}_A \circ \underline{\widetilde{W}}^{a_i,n,T}) Z_T^{a_i,n}\,\mathrm{d}P \\
&= \int_\Omega (\mathbb{1}_A \circ \underline{\widehat{W}}^{a_i,n,T} Z_T^{a_i,n})\,\mathrm{d}P = \int_\Omega (\mathbb{1}_A \circ \underline{\widehat{W}}^{a_i,n,T})\,\mathrm{d}P = \int_{(\underline{\widehat{W}}^{a_i,n,T})^{-1}(A)} \mathrm{d}P = \lambda^{a_i,n}(A),
\end{aligned}$$

so that $\kappa_T^{a_i,n} \approx \lambda^{a_i,n}$ on $\overline{\mathcal{G}}_T^{a_i,n}$. According to Proposition 1.7.4 in Karatzas and Shreve (1998) there exists a unique probability measure $\kappa^{a_i,n} = P^{a_i,n} \circ (\underline{\widetilde{W}}^{a_i,n})^{-1}$ on $\overline{\mathcal{G}}_\infty^{a_i,n}$, such that $\kappa^{a_i,n}$ agrees with $\kappa_T^{a_i,n}$ on $\overline{\mathcal{G}}_T^{a_i,n}$, for each $T < \infty$. In particular, $\underline{\widetilde{W}}^{a_i,n}$ is a Brownian motion under the probability measure $P^{a_i,n}$.

D.3. Obtaining the disagreement via measure changes

To obtain the price dynamics (B.3) from (B.2) we use a density process of the form

$$Z_t^{a_i,n} := \exp\left(\int_0^t \underline{b}(a_i)^\top \Lambda_s^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_s^{a_i,n} - \frac{1}{2}\int_0^t (\Lambda_s^{a_i,n})^2 \underline{b}(a_i)^\top \underline{b}(a_i)\,\mathrm{d}s\right),$$

where $\underline{b}(a_i) \in \mathbb{R}^{3\times 1}$ is a vector function of the form

$$\underline{b}(a_i) = \begin{pmatrix} b_1(a_i) \\ b_2(a_i) \\ b_3(a_i) \end{pmatrix}.$$

By Proposition D.1 and the subsequent extension to an infinite time horizon, the price process S^n in the discrete economy thus satisfies

$$\mathrm{d}S_t^n = \begin{pmatrix} s_G\alpha_G g \\ -s_D\alpha_D \\ s_D\alpha_D - s_G\alpha_G \\ s_G\alpha_\Lambda^n + \hat{\beta}^n(a_i) \end{pmatrix}^\top \begin{pmatrix} 1 \\ D_t \\ \widehat{G}_t^{a_i,n} \\ \Lambda_t^{a_i,n} \end{pmatrix} \mathrm{d}t + \begin{pmatrix} s_D + s_G\Gamma_1^n \\ s_G\Gamma_2^n \\ 0 \end{pmatrix}^\top \boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widetilde{W}}_t^{a_i,n},$$

where we denote

$$\hat{\beta}^n(a_i) := \begin{pmatrix} s_D + s_G\Gamma_1^n & s_G\Gamma_2^n & 0 \end{pmatrix} \boldsymbol{Q}^{a_i,n}\underline{b}(a_i).$$

The $P^{a_i,n}$-Brownian motion $\underline{\widetilde{W}}^{a_i,n} = (\underline{\widetilde{W}}_t^{a_i,n})_{t\geq 0}$ is given by

$$\mathrm{d}\underline{\widetilde{W}}_t^{a_i,n} = \mathrm{d}\underline{\widehat{W}}_t^{a_i,n} - \underline{b}(a_i)\Lambda_s^{a_i,n}\,\mathrm{d}t.$$

Hence, the function $\tilde{\beta}^n(a_i)$ satisfies $\tilde{\beta}^n(a_i) = s_G\alpha_\Lambda^n + \hat{\beta}^n(a_i)$.

Remark D.2. The coefficient $\tilde{\beta}^n$ is the loading of the inference wedge in the drift of the *price*, whereas the belief β^n of the main text is its loading in the drift of the *excess return*. The two are related by

$$\beta^n(a_i) = s_G r + \tilde{\beta}^n(a_i),$$

and likewise $\beta(a_i) = s_G r + \tilde{\beta}(a_i)$ in the limit. The additional term $s_G r$ appears because investors trade the excess return (2.11), which carries the drift term $-rS_t^n\,\mathrm{d}t$ on top of $\mathrm{d}S_t^n$; see (2.12). Neither $\hat{\beta}^n$ nor $\tilde{\beta}^n$ appears in the main text, where individual beliefs are expressed only through β.

When we derive the HJB equation we also have to apply the measure change to the inference wedge $\Lambda^{a_i,n}$. For this purpose, note that under P the inference wedge satisfies (see Corollary C.3)

$$\mathrm{d}\Lambda_t^{a_i,n} = -\alpha_\Lambda^{a_i,n}\Lambda_t^{a_i,n} + \Big(\boldsymbol{\Gamma}^{a_i,n} - \big(\boldsymbol{\Gamma}^n \quad 0\big)\Big)\,\boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widehat{W}}_t^{a_i,n},$$

which under $P^{a_i,n}$ then becomes

$$\mathrm{d}\Lambda_t^{a_i,n} = (\bar{\beta}^n(a_i) - \alpha_\Lambda^{a_i,n})\Lambda_t^{a_i,n} + \Big(\boldsymbol{\Gamma}^{a_i,n} - \big(\boldsymbol{\Gamma}^n \quad 0\big)\Big)\,\boldsymbol{Q}^{a_i,n}\,\mathrm{d}\underline{\widetilde{W}}_t^{a_i,n},$$

where we denote

$$\bar{\beta}^n(a_i) := \Big(\boldsymbol{\Gamma}^{a_i,n} - \big(\boldsymbol{\Gamma}^n \quad 0\big)\Big)\,\boldsymbol{Q}^{a_i,n}\underline{b}(a_i).$$

To avoid confusion and misunderstanding we summarize the different versions of β's in the following list:

Symbol	Origin	Relation
$\hat{\beta}^n$	additional loading of $\Lambda^{a_i,n}$ in drift of S^n under $P^{a_i,n}$	
$\tilde{\beta}^n$	total loading of $\Lambda^{a_i,n}$ in drift of S^n under $P^{a_i,n}$	$\tilde{\beta}^n = s_G\alpha_\Lambda^n + \hat{\beta}^n$
β^n	total loading of $\Lambda^{a_i,n}$ in drift of R^n under $P^{a_i,n}$	$\beta^n = s_G r + \tilde{\beta}^n$
$\bar{\beta}^n$	additional loading of $\Lambda^{a_i,n}$ in drift of $\Lambda^{a_i,n}$ under $P^{a_i,n}$	
β_n^*	total loading of $\Lambda^{a_i,n}$ in drift of R^n under P	$\beta_n^* = \beta^n - \hat{\beta}^n$

E. Utility maximization in the discrete economy

In this appendix, we formulate the individual utility maximization problem in the discrete economy at the level of viscosity solutions. This is consistent with the analysis of HJB equations in the literature (e.g., Fleming and Soner (2006)) and does not assume from the outset that the value function is twice differentiable in the spatial variables.

The outline is as follows: In Section E.1 we introduce the notation for this appendix and provide the standing assumptions. In Section E.2 we derive the HJB equation in the discrete economy, and its pointwise maximizers. In Section E.3 we then give a verification theorem to ensure that these candidates are indeed optimal. In Section E.4 we derive the convergence of the discrete value functions under certain regularity assumptions. Section E.5 is then dedicated to a comparison Lemma required to prove the local uniform convergence of the discrete value functions. In Section E.6 we show the convergence of the first partial derivatives without imposing any new assumptions. Then, in Section E.7 we show local uniform convergence of first and second order partial derivatives of the value functions, which requires one additional assumption. In Section E.8 we derive the limiting HJB equation that describes investors' individual utility maximization problem in the limit, and we provide some additional convergence results. Section E.9 summarizes the assumptions of this appendix.

E.1. Assumptions for this appendix

Throughout this appendix, fix $n \in \mathbb{N}$ and an investor type $a_i \in \Pi_n$. We write

$$\tilde{\rho}^n := -\frac{r\rho}{n}, \qquad m_n(y) := \tilde{\rho}^n + \big(s_G r + \tilde{\beta}^n(a_i)\big)y, \qquad b_n(y) := \big(\bar{\beta}^n(a_i) - \alpha^n_\Lambda\big)y,$$

and we further denote

$$\begin{aligned}
\eta^{a_i,n}_{VV} &:= \begin{pmatrix} s_D + s_G\Gamma^n_1 & s_G\Gamma^n_2 & 0\end{pmatrix} \boldsymbol{Q}^{a_i,n}\boldsymbol{Q}^{a_i,n} \begin{pmatrix} s_D + s_G\Gamma^n_1 & s_G\Gamma^n_2 & 0\end{pmatrix}^\intercal, \\
\eta^{a_i,n}_{V\Lambda} &:= \begin{pmatrix} s_D + s_G\Gamma^n_1 & s_G\Gamma^n_2 & 0\end{pmatrix} \boldsymbol{Q}^{a_i,n}\boldsymbol{Q}^{a_i,n} \left(\boldsymbol{\Gamma}^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}^n & 0\end{pmatrix}\right)^\intercal, \\
\eta^{a_i,n}_{\Lambda\Lambda} &:= \left(\boldsymbol{\Gamma}^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}^n & 0\end{pmatrix}\right) \boldsymbol{Q}^{a_i,n}\boldsymbol{Q}^{a_i,n} \left(\boldsymbol{\Gamma}^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}^n & 0\end{pmatrix}\right)^\intercal
\end{aligned} \tag{E.1}$$

Recall that the investor's wealth process satisfies (2.14) and that the investor solves the optimization problem (2.15). We denote by

$$J_n(t,x,y) = \mathrm{e}^{-rt}\, J^{a_i,n}(t,x,y)$$

the associated value function when the current state is $(V^{a_i,n}_t, \Lambda^{a_i,n}_t) = (x,y)$, and when the value function is normalized at time 0, i.e., $J_n(t,x,y) = \mathrm{e}^{-rt}\, J^{a_i,n}(t,x,y)$ with $J^{a_i,n}$ the value function associated with (2.15). Since e^{-rt} is deterministic and strictly positive, the two problems have the same admissible set and the same optimizers.

Standing assumptions for Appendix E. We shall work under the following assumptions.

(A.E.1) *For the fixed investor type a_i, the utility function $U(a_i,\cdot) : \mathbb{R} \to \mathbb{R}$ is of class C^2, strictly increasing, and strictly concave. Moreover, for every admissible control pair $(c^{a_i,n}, H^{a_i,n}) \in \mathcal{A}(v_0(a_i))$, the discounted utility functional in* (2.15) *is well defined and finite.*

(A.E.2) *Under $P^{a_i,n}$, the state process $X_t^n := (V_t^{a_i,n}, \Lambda_t^{a_i,n})$ is a time-inhomogeneous controlled Markov diffusion on $E := (0,\infty) \times \mathbb{R}$ with control process $(c_t, H_t)_{t\geq 0} \in \mathbb{R} \times \mathbb{R}$, and the dynamic programming principle holds for bounded stopping times. In particular, for each $T > 0$ and every initial condition $(t,x,y) \in [0,T] \times E$, admissible controls may be concatenated at bounded stopping times.*

(A.E.3) *The value function J_n is finite and continuous on $[0,\infty) \times E$.*

(A.E.4) *Whenever we invoke the explicit feedback formulas below, we additionally assume that $U'(a_i,\cdot) : \mathbb{R} \to (0,\infty)$ is onto, so that $(U'(a_i,\cdot))^{-1}$ is well defined on $(0,\infty)$.*

E.2. Statement of the discrete HJB equations

Proposition E.1. *Let Assumptions (A.E.1)–(A.E.3) hold true. Then the value function J_n is a continuous viscosity solution of the HJB equation*

$$\begin{aligned} 0 = &-J_{n,t}(t,x,y) - rx\, J_{n,x}(t,x,y) - b_n(y)\, J_{n,y}(t,x,y) - \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2} J_{n,yy}(t,x,y) \\ &- \sup_{c\in\mathbb{R}} \Big(\mathrm{e}^{-rt}\, U(a_i,c) - c\, J_{n,x}(t,x,y) \Big) \\ &- \sup_{h\in\mathbb{R}} \Big(h\, m_n(y)\, J_{n,x}(t,x,y) + h\, \eta_{V\Lambda}^{a_i,n} J_{n,xy}(t,x,y) + \frac{1}{2} h^2 \eta_{VV}^{a_i,n} J_{n,xx}(t,x,y) \Big), \end{aligned} \tag{E.2}$$

on $[0,\infty) \times E$.

If, in addition, Assumption (A.E.4) holds and J_n is a classical solution of (E.2) *with*

$$J_n \in C^{1,2}\big([0,\infty) \times E\big), \qquad J_{n,xx}(t,x,y) < 0 \quad \textit{for all } (t,x,y) \in [0,\infty) \times E,$$

then the pointwise maximizers in (E.2) *are given by*

$$c_n^*(t,x,y) = \big(U'(a_i,\cdot)\big)^{-1}\big(\mathrm{e}^{rt}\, J_{n,x}(t,x,y)\big), \tag{E.3}$$

and

$$h_n^*(t,x,y) = -\frac{m_n(y)\, J_{n,x}(t,x,y) + \eta_{V\Lambda}^{a_i,n} J_{n,xy}(t,x,y)}{\eta_{VV}^{a_i,n} J_{n,xx}(t,x,y)}. \tag{E.4}$$

Consequently, J_n also satisfies the reduced HJB equation

$$\begin{aligned} -J_{n,t} = {}& rx\, J_{n,x} + b_n(y)\, J_{n,y} + \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2} J_{n,yy} - \frac{\big(m_n(y)\, J_{n,x} + \eta_{V\Lambda}^{a_i,n} J_{n,xy}\big)^2}{2\eta_{VV}^{a_i,n} J_{n,xx}} \\ &+ \mathrm{e}^{-rt}\, U\Big(a_i, \big(U'(a_i,\cdot)\big)^{-1}(\mathrm{e}^{rt}\, J_{n,x})\Big) - \big(U'(a_i,\cdot)\big)^{-1}(\mathrm{e}^{rt}\, J_{n,x})\, J_{n,x}. \end{aligned} \tag{E.5}$$

Proof. By Assumptions (A.E.1)–(A.E.3), the optimization problem (2.15) is a well-posed infinite-horizon controlled Markov diffusion problem, the value function is finite and continuous, and the dynamic programming principle holds for bounded stopping times. Hence, by the standard dynamic-programming-to-viscosity-solution result for controlled diffusions, J_n is a viscosity solution of the associated Bellman equation; see Fleming and Soner 2006, Chapter II, Theorem 5.1. This yields (E.2).

Now assume in addition that J_n is classical and $J_{n,xx} < 0$ everywhere. Then the Hamiltonian in the control variable h is strictly concave, and the first-order condition gives

$$0 = m_n(y)\, J_{n,x}(t,x,y) + \eta_{V\Lambda}^{a_i,n} J_{n,xy}(t,x,y) + h\, \eta_{VV}^{a_i,n} J_{n,xx}(t,x,y),$$

which yields (E.4). Likewise, by strict concavity of $U(a_i,\cdot)$ and the surjectivity of $U'(a_i,\cdot)$, the first-order condition for the consumption Hamiltonian is

$$e^{-rt}U'(a_i,c) - J_{n,x}(t,x,y) = 0,$$

which yields (E.3). Substituting the two maximizers into (E.2) gives (E.5). ■

Remark E.2. Proposition E.1 should be read in two layers. The first part is the viscosity-level result that is natural for the control problem (2.15). The second part is a smooth-case consequence which recovers the feedback formulas used formally in the main text. In particular, the explicit portfolio formula (E.4) requires pointwise existence of $J_{n,xx}$ and $J_{n,xy}$ and strict negativity of $J_{n,xx}$, and therefore should not be interpreted as part of the viscosity statement itself.

E.3. A verification theorem

For simplicity, let us define the processes $M = (M_t)_{t\geq 0}$ and $N = (N_t)_{t\geq 0}$ via

$$\mathrm{d}M_t = \begin{pmatrix} s_D + s_G\Gamma_1^n & s_G\Gamma_2^n & 0 \end{pmatrix} \boldsymbol{Q}^{a_i,n}\, \mathrm{d}\underline{\widetilde{W}}_t^{a_i,n}, \qquad \mathrm{d}N_t = \left(\boldsymbol{\Gamma}^{a_i,n} - \begin{pmatrix} \boldsymbol{\Gamma}^n & 0 \end{pmatrix}\right) \boldsymbol{Q}^{a_i,n}\, \mathrm{d}\underline{\widetilde{W}}_t^{a_i,n},$$

so that $\langle M,M\rangle_t = \eta_{VV}^{a_i,n}\, t$, $\langle M,N\rangle = \eta_{V\Lambda}^{a_i,n}\, t$, and $\langle N,N\rangle_t = \eta_{\Lambda\Lambda}^{a_i,n}\, t$. The dynamics of the processes $V^{a_i,n}$ and $\Lambda^{a_i,n}$ can then be written as

$$\mathrm{d}V_t^{a_i,n} = (rV_t^{a_i,n} - c_t^{a_i,n} + H_t^{a_i,n} m_n(t,\Lambda_t^{a_i,n}))\,\mathrm{d}t + H^{a_i,n}\,\mathrm{d}M_t, \qquad \mathrm{d}\Lambda_t^{a_i,n} = b_n(\Lambda_t^{a_i,n})\,\mathrm{d}t + \mathrm{d}N_t.$$

Now, consider a function $\varphi \in C^{1,2}([0,\infty)\times E)$ and let $c,h\in\mathbb{R}$. Define the infinitesimal generator $\mathcal{L}^{c,h}$ of the system governed by $V^{a_i,n}$ and $\Lambda^{a_i,n}$, i.e.,

$$\mathcal{L}^{c,h}\varphi(t,x,y) := (rx - c + hm_n(y))\,\varphi_x + b_n(y)\,\varphi_y + h\,\eta_{V\Lambda}^{a_i,n}\,\varphi_{xy} + \frac{1}{2}h^2\eta_{VV}^{a_i,n}\,\varphi_{xx} + \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2}\,\varphi_{yy},$$

so that (E.2) reads as $-J_{n,t} = \sup_{c,h}\left[\mathcal{L}^{c,h}J_n + e^{-rt}U(a_i,c)\right]$. For a fixed $W \in C^{1,2}([0,\infty)\times E)$ consider the following assumptions:

(A.E.5) *Let $c_n^*(t,x,y)$ and $h_n^*(t,x,y)$ denote the maximizers of* (E.2) *associated to W given by equations* (E.3) *and* (E.4)*, respectively, with J_n replaced by W. The stochastic differential equation*

$$\mathrm{d}V_s^* = (rV_s^* - c_n^*(s,V_s^*,\Lambda_s) + h_n^*(s,V_s^*,\Lambda_s)m_n(s,\Lambda_s))\,\mathrm{d}s + h_n^*(s,V_s^*,\Lambda_s)\,\mathrm{d}M_s, \qquad V_t^* = x,$$

alongside $\mathrm{d}\Lambda_S = b_n(\Lambda_s)\,\mathrm{d}s + \mathrm{d}N_s$, $\Lambda_t = y$, admits for every $(t,x,y) \in [0,\infty) \times E$ a unique strong solution $V^* = (V_s^*)_{s\geq t}$ with $V_s^* > 0$ $P^{a_i,n}$-a.s for every $s \geq t$, and $(c_n^*, H_n^*) \in \mathcal{A}(v_0(a_i))$ where $(c_n^*, H_n^*) \coloneqq (c_n^*(s,V_s^*,\Lambda_s), h_n^*(s,V_s^*,\Lambda_s))_{s\geq t}$.

(A.E.6) *For every $(t,x,y) \in [0,\infty) \times E$, every $(c,H) \in \mathcal{A}(v_0(a_i))$ with induced state process (V,Λ) started at (t,x,y), and every $T > t$, define the stopping time*

$$\tau_k \coloneqq \inf\{s \geq t \colon V_s \notin (1/k, k) \text{ or } |\Lambda_s| \geq k\} \wedge T.$$

Then, the following two assertions hold true:

(i) *both stochastic integrals*

$$\left(\int_t^{s\wedge\tau_k} W_x(u,V_u,\Lambda_u)H_u\,\mathrm{d}M_u\right)_{s\geq t}, \qquad \left(\int_t^{s\wedge\tau_k} W_y(u,V_u,\Lambda_u)\,\mathrm{d}N_u)\right)_{s\geq t}$$

are true martingales for each $k \in \mathbb{N}$;

(ii) *and $\tau_k \to T$ $P^{a_i,n}$-a.s. as $k \to \infty$, and the family $(W(\tau_k, V_{\tau_k}, \Lambda_{\tau_k}))_{k\in\mathbb{N}}$ is uniformly integrable.*

(A.E.7) *For every $(t,x,y) \in [0,\infty) \times E$ it holds that $\liminf_{T\to\infty} \mathrm{E}[W(T,V_T,\Lambda_T)] \geq 0$ for every $(c,H) \in \mathcal{A}(v_0(a_i))$ with the induced state process (V,Λ) started at (t,x,y), and with $\lim_{T\to\infty} \mathrm{E}[W(T,V_T^*,\Lambda_T)] = 0$ along the path where (c_n^*, H_n^*) satisfies Assumption (A.E.5).*

Proposition E.3. *Let (A.E.1) and (A.E.4) hold, and assume $W \in C^{1,2}([0,\infty)\times E)$ satisfies:*

(i) *$W_x(t,x,y) > 0$ and $W_{xx}(t,x,y) < 0$ for every $(t,x,y) \in [0,\infty) \times E$;*

(ii) *W is a classical solution to the HJB equation* (E.2) *on $[0,\infty) \times E$, with the supremum in* (E.2) *attained at $c_n^*(t,x,y)$ and $h_n^*(t,x,y)$ defined as in* (E.3) *and* (E.4)*, respectively;*

(iii) *W satisfies Assumptions (A.E.5)–(A.E.7).*

Then, it holds $W(t,x,y) = J_n(t,x,y)$ for every $(t,x,y) \in [0,\infty) \times E$, so that $\mathrm{e}^{rt}\,W(t,x,y)$ coincides with the value function of (2.15) *at state $(V_t^{a_i,n}, \Lambda_t^{a_i,n}) = (x,y)$. Moreover, the feedback pair (c_n^*, H_n^*) defined as in Assumption (A.E.5) is $v_0(a_i)$-admissible and attains the essential supremum in* (2.15)*, which implies that (c_n^*, H_n^*) is optimal.*

Proof. Throughout the proof let us fix $(t, x, y) \in [0, \infty) \times E$. We first note that the HJB equation (E.2) can be written as $W_t(t, x, y) + \sup_{(c,h) \in \mathbb{R}^2} \left[\mathcal{L}^{c,h} W(t, x, y) + e^{-rt} U(a_i, c)\right] = 0$, so that for every $(c, h) \in \mathbb{R}^2$ it holds that

$$W_t(t, x, y) + \mathcal{L}^{c,h} W(t, x, y) + e^{-rt} U(a_i, c) \leq 0, \tag{E.6}$$

with equality in (E.6) if and only if $(c, h) = (c_n^*, h_n^*)$.

Now, let $(c, H) \in \mathcal{A}(v_0(a_i))$ be an arbitrary consumption-trading pair with induced state process (V, Λ), $(V_t, \Lambda_t) = (x, y)$, and fix $T > t$. With τ_k defined as in assumption (A.E.6), the entire trajectory remains inside the compact subset $[1/k, k] \times [-k, k] \subseteq E$ on $[t, \tau_k]$. Hence, we may apply Itô's lemma, so that

$$\begin{aligned} W(\tau_k, V_{\tau_k}, \Lambda_{\tau_k}) = W(t, x, y) &+ \int_t^{\tau_k} (W_t(s, V_s, \Lambda_s) + \mathcal{L}^{c,h} W(s, V_s, \Lambda_s))\, \mathrm{d}s \\ &+ \int_t^{\tau_k} W_x(s, V_s, \Lambda_s) H_s \, \mathrm{d}M_s + \int_t^{\tau_k} W_y(s, V_s, \Lambda_s)\, \mathrm{d}N_s. \end{aligned}$$

From (E.6) we know that $W_t(s, V_s, \Lambda_s) + \mathcal{L}^{c,h} W(s, V_s, \Lambda_s) \leq -e^{-rs} U(a_i, c_s)$, and by Assumption (A.E.6).(i) both stochastic integrals in the above equation are true martingales so that

$$\mathrm{E}[W(\tau_k, V_{\tau_k}, \Lambda_{\tau_k})] + \mathrm{E}\left[\int_t^{\tau_k} e^{-rs} U(a_i, c_s)\, \mathrm{d}s\right] \leq W(t, x, y) \qquad \forall\, k \in \mathbb{N}.$$

By Assumption (A.E.6).(ii) it holds that $\tau_k \to T$ $P^{a_i,n}$-a.s. as $k \to \infty$, which by continuity of W implies $W(\tau_k, V_{\tau_k}, \Lambda_{\tau_k}) \to W(T, V_T, \Lambda_T)$ $P^{a_i,n}$-a.s., and thus in probability. Moreover, also by Assumption (A.E.6).(ii), the family $(W(\tau_k, V_{\tau_k}, \Lambda_{\tau_k}))_{k \in \mathbb{N}}$ is uniformly integrable, so that Theorem 13.7 in Williams (1991) implies $\mathrm{E}[W(\tau_k, V_{\tau_k}, \Lambda_{\tau_k})] \to \mathrm{E}[W(T, V_T, \Lambda_T)]$ as $k \to \infty$. On the other hand, since the utility function U is finite for every control pair $(c, H) \in \mathcal{A}(v_0(a_i))$ by Assumption (A.E.1), it holds that

$$\mathrm{E}\left[\int_t^{\tau_k} e^{-rs} U(a_i, c_s)\, \mathrm{d}s\right] \to \mathrm{E}\left[\int_t^{T} e^{-rs} U(a_i, c_s)\, \mathrm{d}s\right] \qquad \text{as } k \to \infty,$$

and therefore

$$\mathrm{E}[W(T, V_T, \Lambda_T)] + \mathrm{E}\left[\int_t^{T} e^{-rs} U(a_i, c_s)\, \mathrm{d}s\right] \leq W(t, x, y) \qquad \forall\, T > t.$$

Hence, by letting $T \to \infty$ and by using Assumptions (A.E.1) and (A.E.7), we find that

$$\mathrm{E}\left[\int_t^{\infty} e^{-rs} U(a_i, c_s)\, \mathrm{d}s\right] \leq W(t, x, y) - \liminf_{T \to \infty} \mathrm{E}[W(T, V_T, \Lambda_T)] \leq W(t, x, y).$$

Since $(c, H) \in \mathcal{A}(v_0(a_i))$ was arbitrary, we may take the essential supremum over the admissible consumption-trading pairs conditional on $\mathcal{G}_t^{a_i,n}$ on the left-hand side of the above inequality, which eventually gives $J_n(t, x, y) \leq W(t, x, y)$.

To prove the reverse inequality $J_n(t, x, y) \geq W(t, x, y)$ we apply the exact same argumentation as before, but now with the pair $(c_n^*, H_n^*) \in \mathcal{A}(v_0(a_i))$ from Assumption (A.E.5) and with the unique strong solution V^*. Since (E.6) now holds with equality along this trajectory, every inequality in the previous step above is an equality. More precisely, it holds that

$$\mathrm{E}[W(T, V_T^*, \Lambda_T)] + \mathrm{E}\left[\int_t^T e^{-rs} U(a_i, c_s^*(s, V_s^*, \Lambda_s))\,\mathrm{d}s\right] = W(t, x, y) \qquad \forall\, T > t.$$

Hence, by letting $T \to \infty$ Assumption (A.E.7) implies

$$\mathrm{E}\left[\int_t^\infty e^{-rs} U(a_i, c_s^*(s, V_s^*, \Lambda_s))\,\mathrm{d}s\right] = W(t, x, y).$$

Now, the left-hand side of the above identity is one particular admissible value of the conditional expectation whose essential supremum defines $J_n(t, x, y)$, and therefore it holds that $J_n(t, x, y) \geq W(t, x, y)$. Hence, we have proven that $J_n(t, x, y) = W(t, x, y)$, and in particular by this last step the optimal value is attained at (c_n^*, H_n^*). This concludes the proof. ■

E.4. Convergence of the discrete HJB equations

We now formulate a convergence result for the discrete HJB equations in the viscosity sense. The proof uses the half-relaxed-limits method and therefore does not require an explicit representation of the value functions $J_n = \mathrm{e}^{-rt}\, J^{a_i,n}$.

By setting

$$a_n := \eta_{\Lambda\Lambda}^{a_i,n}, \qquad d_n := \eta_{V\Lambda}^{a_i,n}, \qquad \sigma_n := \eta_{VV}^{a_i,n},$$

the Bellman operator corresponding to (E.2) is

$$\begin{aligned} F_n(t, x, y, q, p, X) := &-q - rx\, p_x - b_n(y)\, p_y - \frac{a_n}{2} X_{yy} - \sup_{c\in\mathbb{R}} \Big(\mathrm{e}^{-rt}\, U(a_i, c) - c\, p_x\Big) \\ &- \sup_{h\in\mathbb{R}} \Big(h\, m_n(y)\, p_x + h\, d_n\, X_{xy} + \frac{1}{2} h^2 \sigma_n\, X_{xx}\Big), \end{aligned} \tag{E.7}$$

for $(t, x, y) \in [0, \infty) \times (0, \infty) \times \mathbb{R}$, $q \in \mathbb{R}$, $p = (p_x, p_y) \in \mathbb{R}^2$, and $X = \begin{pmatrix} X_{xx} & X_{xy} \\ X_{xy} & X_{yy} \end{pmatrix} \in \mathbb{S}^2$. For simplicity, let us further denote the *consumption Hamiltonian*

$$C_n(t, p) := \sup_{c\in\mathbb{R}} \big(\mathrm{e}^{-rt}\, U(a_i, c) - c\, p_x\big),$$

and the *demand Hamiltonian*

$$H_n(t,y,p,X) \coloneqq \sup_{h\in\mathbb{R}} \Big(h\, m_n(y)\, p_x + h\, d_n\, X_{xy} + \tfrac{1}{2} h^2 \sigma_n\, X_{xx} \Big),$$

so that the Bellman operator $F_n(\cdot)$ from (E.7) reads as

$$F_n(t,x,y,q,p,X) = -q - rx\, p_x - b_n(y)\, p_y - \frac{a_n}{2} X_{yy} - C_n(t,p) - H_n(t,y,p,X).$$

We denote the lower- and upper semicontinuous envelope of F_n by $(F_n)_*$ and F_n^*, respectively, that is

$$\begin{aligned} (F_n)_*(t,x,y,q,p,X) &\coloneqq \sup\big\{G(t,x,y,q,p,X) \colon G \text{ is lower semicontinuous and } G \le F_n\big\}, \\ F_n^*(t,x,y,q,p,X) &\coloneqq \inf\big\{G(t,x,y,q,p,X) \colon G \text{ is upper semicontinuous and } G \ge F_n\big\}, \end{aligned}$$

and we use the same convention for the limiting operator F by writing F_* and F^*.

Lemma E.4. *The function $(p,X) \mapsto H_n(t,y,p,X)$ is convex and lower semicontinuous, real-valued and continuous on $\{X_{xx} < 0\}$, and its upper semicontinuous envelope H_n^* is equal to $+\infty$ on $\{X_{xx} \ge 0\}$. Moreover, it holds that*

$$H_n(t,y,p,X) = \begin{cases} -\frac{(m_n(y)\, p_x + d_n\, X_{xy})^2}{2\sigma_n X_{xx}}, & X_{xx} < 0, \\ 0, & X_{xx} = 0,\ m_n(y)\, p_x + d_n\, X_{xy} = 0, \\ +\infty, & \textit{otherwise}, \end{cases}$$

with $H_n(t,y,p,X) \in [0,\infty)$ on $\{X_{xx} < 0\}$.

Proof. The first order condition reads as $m_n(y)\, p_x + d_n\, X_{xy} + h\sigma_n\, X_{xx} = 0$, and the second order condition is given by $\sigma_n\, X_{xx} < 0$. Since $\sigma_n > 0$ the supremum is attained at

$$h^* = -\frac{m_n(y)\, p_x + d_n\, X_{xy}}{\sigma_n\, X_{xx}}$$

whenever $X_{xx} < 0$, so that $H_n(t,y,p,X) = -\frac{(m_n(y)p_x + d_n X_{xy})^2}{2\sigma_n X_{xx}}$ on the set $\{X_{xx} < 0\}$. Moreover, if $X_{xx} > 0$ or $X_{xx} = 0$ and $m_n(y)\, p_x + d_n\, X_{xy} \ne 0$, then $H_n(t,x,y,p,X) = +\infty$, and if $X_{xx} = m_n(y)\, p_x + d_n\, X_{xy} = 0$ it follows that $H_n(t,y,p,X) = 0$. Then, it follows immediately that $(p,X) \mapsto H_n(t,y,p,X)$ is continuous and real-valued on $\{X_{xx} < 0\}$. Convexity of $(p,X) \mapsto H_n(t,y,p,X)$ follows from the fact that $H_n(t,y,p,X)$ is the supremum of a family of affine functions in (p,X) indexed by h, and lower semicontinuity follows from the very same fact and Theorem 12.1 in Rockafellar (1970). The statement about the upper semicontinuous envelope follows from $H_n(t,y,p,X) = +\infty$ on $\{X_{xx} > 0\}$, so that the supremum on an open neighbourhood at $X_{xx} = 0$ is $+\infty$. This concludes the proof. ∎

Lemma E.5. *The function $p \mapsto C_n(t,p)$ is convex, lower semicontinuous, continuous and finite on $\{p_x > 0\}$, and $+\infty$ on $\{p_x < 0\}$.*

Proof. Convexity and lower semicontinuity follow as in the proof of Lemma E.4 from the fact that $p \mapsto C_n(t,p)$ is the supremum of affine functions, and together with Theorem 12.1 in Rockafellar (1970). Finiteness on $\{p_x > 0\}$ follows from assumption (A.E.1), and continuity on $\{p_x > 0\}$ follows from the fact that a finite convex function on an open interval is continuous (see Theorem 10.1 together with Corollary 10.1.1 in Rockafellar (1970)). ■

Corollary E.6. *The function $(q,p,X) \mapsto F_n(t,x,y,q,p,X)$ as defined in* (E.7) *is upper semicontinuous and therefore satisfies $F_n = F_n^*$. Its lower semicontinuous envelope satisfies*

$$(F_n)_*(t,x,y,q,p,X) = \begin{cases} F_n(t,x,y,q,p,X) & \text{if } p_x > 0 \text{ and } X_{xx} < 0, \\ -\infty & \text{otherwise,} \end{cases}$$

and the same holds true for the limiting operator F.

Proof. We write (E.7) in the form

$$F_n(t,x,y,q,p,X) = -q - rx\, p_x - b_n(y)\, p_y - \frac{a_n}{2} X_{yy} - C_n(t,p) - H_n(t,y,p,X).$$

Since $(q,p,X) \mapsto -q - rx\,p_x - b_n(y)\,p_y - \frac{a_n}{2} X_{yy}$ is continuous it is both upper and lower semicontinuous. Further, the functions $(p,X) \mapsto -H_n(t,y,p,X)$ and $p \mapsto -C_n(t,p)$ are upper semicontinuous because they are the negative of lower semicontinuous functions, which follows from Lemma E.4 and Lemma E.5, respectively. Hence $(q,p,X) \mapsto F_n(t,x,y,q,p,X)$ is upper semicontinuous as a sum of upper semicontinuous functions. The fact that the lower semicontinuous envelope satisfies $(F_n)_* = -\infty$ outside of $\{p_x > 0\} \cap \{X_{xx} < 0\}$ follows from the fact that the upper semicontinuous envelope of $(p,X) \mapsto H_n(t,y,p,X)$ is $+\infty$ on $\{X_{xx} \geq 0\}$ by Lemma E.4, and the same argumentation holds for $p \mapsto C_n(t,p)$ on $\{p_x \leq 0\}$ which is a consequence of Lemma E.5. ■

Throughout, the limiting equation is understood in the sense of discontinuous viscosity solutions (see Crandall, Ishii, and Lions (1992)): a locally bounded upper semicontinuous function u is a *subsolution* if $F_*(t,x,y,\phi_t,D\phi,D^2\phi) \leq 0$ at every point where $u - \phi$ has a local maximum, and a locally bounded lower semicontinuous function v is a *supersolution* if $F^* = F \geq 0$ at every point where $v - \psi$ has a local minimum. By Corollary E.6 the subsolution requirement is void unless $\phi_x > 0$ and $\phi_{xx} < 0$. The supersolution requirement is unchanged.

Theorem E.7. *Let Assumptions (A.E.1)–(A.E.3) hold true, and assume the following conditions are satisfied:*

(A.E.8) *There exist continuous functions $b : [0,\infty) \times E \to \mathbb{R}$ and $m : [0,\infty) \times E \to \mathbb{R}$, and constants $a \geq 0$, $d \in \mathbb{R}$, and $\sigma > 0$, such that for every $T > 0$ and every compact interval $I_y \subset \mathbb{R}$, the following assertions hold true:*

$$\sup_{y \in I_y} |b_n(y) - b(\cdot, y)| \longrightarrow 0, \tag{E.8}$$

$$\sup_{(t,y)\in[0,T]\times I_y} |m_n(y) - m(t,\cdot,y)| \longrightarrow 0, \tag{E.9}$$

$$a_n \to a, \qquad d_n \to d, \qquad \sigma_n \to \sigma. \tag{E.10}$$

The limiting coefficients further satisfy $d = 0$, the maps b and m are locally Lipschitz in y, uniformly in t on compacts, and $2b(t,y)y \leq ry^2$ for all (t,y).

(A.E.9) *There exists $\underline{\sigma} > 0$ such that $\sigma_n = \eta_{VV}^{a_i,n} \geq \underline{\sigma}$ for all $n \in \mathbb{N}$.*

(A.E.10) *There is $\kappa \in [0,2)$ such that for every compact interval $I_x \subset (0,\infty)$,*

$$\sup_{n\geq 1} \sup_{(t,x,y)\in[0,\infty)\times I_x\times\mathbb{R}} \frac{\mathrm{e}^{rt}\left|J_n(t,x,y)\right|}{\varrho(y)} \;<\; \infty, \qquad \varrho(y) = 1 + |y|^\kappa.$$

(A.E.11) *For every $T > 0$ and every compact set $K \subset (0,\infty)\times\mathbb{R}$, there exist $\lambda_{T,K} > 0$ and $c_{T,K} > 0$ such that for every $n \in \mathbb{N}$ and every (t,y) with $(t,x,y) \in [0,T]\times K$ for some x, the map*

$$x \mapsto J_n(t,x,y) + \frac{\lambda_{T,K}}{2}x^2$$

is concave on each x-section of K, and the map

$$x \mapsto J_n(t,x,y) - c_{T,K}x$$

is non-decreasing on each x-section of K.

(A.E.12) *Let ϱ be as in Assumption (A.E.10), and assume that $\overline{J}$ and $\underline{J}$, defined via*

$$\overline{J}(t,x,y) \coloneqq \lim_{\varepsilon\searrow 0} \sup\Big\{J_n(s,\xi,\eta) : n \geq \varepsilon^{-1},\, |s-t| + |\xi - x| + |\eta - y| < \varepsilon\Big\},$$
$$\underline{J}(t,x,y) \coloneqq \lim_{\varepsilon\searrow 0} \inf\Big\{J_n(s,\xi,\eta) : n \geq \varepsilon^{-1},\, |s-t| + |\xi - x| + |\eta - y| < \varepsilon\Big\},$$

satisfy

$$\lim_{x\searrow 0} \sup_{(t,y)\in[0,\infty)\times\mathbb{R}} \frac{\mathrm{e}^{rt}(\overline{J} - \underline{J})^+(t,x,y)}{\varrho(y)} = 0, \qquad \lim_{x\to\infty} \sup_{(t,y)\in[0,\infty)\times\mathbb{R}} \frac{\mathrm{e}^{rt}(\overline{J} - \underline{J})^+(t,x,y)}{\varrho(y)} = 0.$$

Then there exists a unique continuous viscosity solution

$$J : [0, \infty) \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$$

of the limiting HJB equation

$$F(t, x, y, J_t, DJ, D^2 J) = 0,$$

which is obtained from (E.7) *by replacing* $(b_n, m_n, a_n, d_n, \sigma_n)$ *with* (b, m, a, d, σ)*, and*

$$J_n \to J \qquad \textit{locally uniformly on } [0, \infty) \times (0, \infty) \times \mathbb{R}.$$

To eventually prove Theorem E.7 we first need to establish some auxiliary results. For a function $u\colon \mathbb{R}^n \to \mathbb{R}$, let us denote the *second-order subjet* of u at x by $\mathcal{J}^{2,-}u(x)$, that is the set of pairs $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ such that

$$u(y) \geq u(x) + p^\mathsf{T}(y - x) + \frac{1}{2}(y - x)^\mathsf{T} X (y - x) + o(\|y - x\|^2) \qquad \text{as } y \to x.$$

Lemma E.8. *Let Assumption (A.E.11) hold true. If* $(p, X) \in \mathcal{J}^{2,-} J_n(t_0, x_0, y_0)$ *for some* $(t_0, x_0, y_0) \in [0, T] \times K$*, then* $X_{xx} \leq -\lambda_{T,K}$*, and* $p_x \geq c_{T,K} > 0$*.*

Proof. By Assumption (A.E.11) the function $g(x) = J_n(t_0, x, y_0) + \frac{\lambda_{T,K}}{2} x^2$ is concave. Let $q \in \mathbb{R}$ be a supergradient of g at x_0, so that $g(x) \leq g(x_0) + q(x - x_0)$ near x_0. Now, the second-order subjet expansion restricted to the x-line yields

$$J_n(t_0, x, y_0) \geq J_n(t_0, x_0, y_0) + p_x(x - x_0) + \frac{1}{2}(x - x_0)^2 X_{xx} + o(|x - x_0|^2).$$

Then, by substituting $J_n = g - \frac{\lambda_{T,K}}{2} x^2$ into this expansion, and by taking $x = x_0 \pm h$, we find that

$$\pm h(q - \lambda_{T,K} x_0) - \frac{\lambda_{T,K}}{2} h^2 \geq \pm h\, p_x + \frac{1}{2} h^2 X_{xx} + o(h^2).$$

By summing up the two inequalities, i.e., the "+" and the "−" inequality, and by dividing by h^2 yields $-\lambda_{T,K} \geq X_{xx} + 2o(h^2)/h^2 \geq X_{xx}$.

For the second assertion, note that since $x \mapsto J_n(t_0, x, y_0) - c_{T,K} x$ is non-decreasing by Assumption (A.E.11), we have for each $h < 0$ that $J_n(t_0, x_0 + h, y_0) - J_n(t_0, x_0, y_0) \leq c_{T,K} h$. The subjet expansion restricted to the x-line then gives

$$J_n(t_0, x_0 + h, y_0) - J_n(t_0, x_0, y_0) \geq p_x h + \frac{1}{2} X_{xx} h^2 + o(h^2).$$

Hence, we have $p_x h + \frac{1}{2} X_{xx} h^2 + o(h^2) \leq c_{T,K} h$, so that dividing by $h < 0$ and letting $h \nearrow 0$ yields $p_x \geq c_{T,K}$. This concludes the proof. ■

Lemma E.9. *Under Assumptions (A.E.10) and (A.E.11) it holds that for every $T > 0$ and every compact subset $K' \subset \mathrm{int}(K)$ there exists $L_{T,K,K'} < \infty$ with*

$$|J_n(t,x,y) - J_n(t,x',y)| \le L_{T,K,K'}|x - x'|$$

for all $n \in \mathbb{N}$ and all $(t,x,y),(t,x',y) \in [0,T] \times K'$.

Proof. A concave function bounded on a convex compact set is Lipschitz on any interior compact subset, and its Lipschitz constant depends only on the bound of the function and on the distance between K' and ∂K. Indeed, let g be concave on K, and assume that $|g| \le M$ on K. Since $-g$ is convex, the local Lipschitz estimate follows from Theorem 10.4 in Rockafellar (1970) and its proof. Applying this fact to $g\colon x \mapsto J_n(t,x,y) + \frac{\lambda_{T,K}}{2}x^2$, which is uniformly bounded by Assumption (A.E.10) yields

$$\begin{aligned} |J_n(t,x,y) - J_n(t,x',y)| &\le |g(x) - g(x')| + \frac{\lambda_{T,K}}{2}|x^2 - (x')^2| \\ &\le \Big(\tilde{L}_{K,K'} + \lambda_{T,K} \sup_{x \in K}|x|\Big)|x - x'|, \end{aligned}$$

which concludes the proof. ■

Lemma E.10. *Let Assumptions (A.E.10) and (A.E.11) hold, and let*

$$\underline{J}(t,x,y) := \lim_{\varepsilon \searrow 0} \inf\big\{J_n(s,\xi,\eta) : n \ge \varepsilon^{-1}, |s-t| + |\xi - x| + |\eta - y| < \varepsilon\big\}.$$

Then for every (t,y) such that $(t,x,y) \in [0,T] \times \mathrm{int}(K)$, the map $x \mapsto \underline{J}(t,x,y) + \frac{\lambda_{T,K}}{2}x^2$ is concave on each x-section of $\mathrm{int}(K)$, and the map $x \mapsto \underline{J}(t,x,y) - c_{T,K}x$ is non-decreasing on each x-section of $\mathrm{int}(K)$.

Proof. Let $L_{T,K,K'}$ be the Lipschitz constant from Lemma E.9. For $\varepsilon > 0$ set

$$I_\varepsilon(t,x,y) := \inf\big\{J_n(s,x,\eta) : n \ge \varepsilon^{-1}, |s-t| + |\eta - y| < \varepsilon\big\}.$$

From Lemma E.9 it follows that replacing x by any ξ with $|\xi - x| < \varepsilon$ changes $J_n(s,\xi,\eta)$ by at most $\varepsilon\, L_{T,K,K'}$, and thus by definition $\big|\underline{J}(t,x,y) - \lim_{\varepsilon\searrow 0} I_\varepsilon(t,x,y)\big| = 0$. Now, for a fixed (s,η) and n, the map $x \mapsto J_n(s,x,\eta) + \frac{\lambda_{T,K}}{2}x^2$ is concave by Assumption (A.E.11), and because an infimum of concave functions is concave, it follows that $x \mapsto I_\varepsilon(t,x,y) + \frac{\lambda_{T,K}}{2}x^2$ is concave for each ε, and so is its pointwise limit as $\varepsilon \searrow 0$.

On the other hand, for a fixed (s,η) and n the map $x \mapsto J_n(s,x,\eta) - c_{T,K}x$ is non-decreasing by Assumption (A.E.11) because an infimum of non-decreasing functions is non-decreasing. Hence the map $x \mapsto I_\varepsilon(t,x,y) - c_{T,K}x$ is non-decreasing for each ε, which eventually implies that its pointwise limit as $\varepsilon \searrow 0$ is non-decreasing. ■

Lemma E.11. *Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution, and assume that $x \mapsto v(t,x,y) + \frac{\lambda_{T,K}}{2}x^2$ is concave. Let $(p, X) \in \mathcal{J}^{2,+}u(t,x,y)$ and $(p, Y) \in \mathcal{J}^{2-}v(t',x',y')$ with $X \leq Y$ (i.e., $Y - X$ is positive semi-definite). Then $X_{xx} \leq Y_{xx} \leq -\lambda_{T,K} < 0$.*

Proof. Since $Y - X$ is positive semi-definite it holds that $\underline{a}^\mathsf{T}(Y - X)\underline{a} \geq 0$ for every vector $\underline{a} \in \mathbb{R}^2$, so that

$$\begin{pmatrix}1 & 0\end{pmatrix}(Y - X)\begin{pmatrix}1\\0\end{pmatrix} = Y_{xx} - X_{xx} \geq 0,$$

and therefore $X_{xx} \leq Y_{xx}$. The fact that $Y_{xx} \leq -\lambda_{T,K}$ follows from Lemma E.8. ∎

With these results in hand we are now ready to give the proof to Theorem E.7.

Proof of Theorem E.7. Fix $T > 0$ and a compact set $K \subset (0,\infty) \times \mathbb{R}$. We work on $[0,T] \times K$.

Step 1: definition of the half-relaxed limits. By Assumption (A.E.10), the upper and lower half-relaxed limits of the sequence (J_n) are well defined:

$$\overline{J}(t,x,y) := \limsup_{n\to\infty}{}^* J_n(t,x,y), \qquad \underline{J}(t,x,y) := \liminf_{n\to\infty}{}_* J_n(t,x,y),$$

that is,

$$\overline{J}(t,x,y) = \lim_{\varepsilon\downarrow 0} \sup\Big\{ J_n(s,\xi,\eta) : n \geq \varepsilon^{-1},\ |s-t| + |\xi - x| + |\eta - y| < \varepsilon \Big\},$$

and

$$\underline{J}(t,x,y) = \lim_{\varepsilon\downarrow 0} \inf\Big\{ J_n(s,\xi,\eta) : n \geq \varepsilon^{-1},\ |s-t| + |\xi - x| + |\eta - y| < \varepsilon \Big\}.$$

The function $\overline{J}$ is upper semicontinuous, the function $\underline{J}$ is lower semicontinuous, and

$$\underline{J} \leq \overline{J}.$$

Step 2: $\overline{J}$ is a viscosity subsolution of the limiting equation. Let $\phi \in C^{1,2}$ be such that $\overline{J} - \phi$ attains a strict local maximum at $(t_0,x_0,y_0) \in (0,T) \times \mathrm{int}(K)$ and

$$(\overline{J} - \phi)(t_0,x_0,y_0) = 0.$$

By the standard half-relaxed-limits argument of Crandall, Ishii, and Lions 1992, Lemma 6.1 and Remark 6.3, there exist integers $n_k \to \infty$ and points

$$(t_k,x_k,y_k) \to (t_0,x_0,y_0)$$

such that

$$J_{n_k}(t_k, x_k, y_k) \to \overline{J}(t_0, x_0, y_0),$$

and $J_{n_k} - \phi$ attains a local maximum at (t_k, x_k, y_k).

Since each J_{n_k} is a viscosity subsolution of

$$F_{n_k}(t, x, y, J_{n_k,t}, DJ_{n_k}, D^2 J_{n_k}) = 0,$$

we have

$$F_{n_k}\big(t_k, x_k, y_k, \phi_t(t_k, x_k, y_k), D\phi(t_k, x_k, y_k), D^2\phi(t_k, x_k, y_k)\big) \leq 0.$$

On $\{\phi_x(t_0, x_0, y_0) \leq 0\} \cup \{\phi_{xx}(t_0, x_0, y_0) \geq 0\}$ we know from Corollary E.6 that

$$F_*\big(t_0, x_0, y_0, \phi_t(t_0, x_0, y_0), D\phi(t_0, x_0, y_0), D^2\phi(t_0, x_0, y_0)\big) = -\infty \leq 0,$$

and so there is nothing to prove. So assume without loss of generality that $\phi_x(t_0, x_0, y_0) > 0$ and $\phi_{xx}(t_0, x_0, y_0) < 0$. By assumption $\phi \in C^{1,2}$, so that ϕ_x and ϕ_{xx} are both continuous, and because $(t_k, x_k, y_k) \to (t_0, x_0, y_0)$ there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ it holds that

$$\phi_x(t_k, x_k, y_k) \geq \frac{1}{2}\phi_x(t_0, x_0, y_0) > 0, \qquad \phi_{xx}(t_k, x_k, y_k) \leq \frac{1}{2}\phi_{xx}(t_0, x_0, y_0) < 0.$$

Indeed, for all $\delta > 0$ there is $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$: $\|(t_k, x_k, y_k) - (t_0, x_0, y_0)\| < \delta$. Moreover, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\|(t_k, x_k, y_k) - (t_0, x_0, y_0)\| < \delta$ it holds $|\phi_x(t_k, x_k, y_k) - \phi_x(t_0, x_0, y_0)| < \varepsilon$. Hence, for a fixed $\varepsilon > 0$ we find $\delta > 0$, and for this particular $\delta > 0$ we find $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ it holds $\|(t_k, x_k, y_k) - (t_0, x_0, y_0)\| < \delta$ and thus $\phi_x(t_k, x_k, y_k) - \phi_x(t_0, x_0, y_0) \in (-\varepsilon, +\varepsilon)$. Moreover, since $\phi_x(t_0, x_0, y_0) > 0$ there exists $\varepsilon' > 0$ so that $\phi_x(t_0, x_0, y_0) \geq \varepsilon'$, and by taking $\varepsilon = \frac{\varepsilon'}{2}$ we find that

$$\phi_x(t_k, x_k, y_k) - \phi_x(t_0, x_0, y_0) + \frac{1}{2}\phi_x(t_0, x_0, y_0) > -\frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = 0,$$

which suffices to conclude $\phi_x(t_k, x_k, y_k) \geq \frac{1}{2}\phi_x(t_0, x_0, y_0)$. The second inequality follows by a similar argument. On the region $\{\phi_x(t_0, x_0, y_0) > 0\} \cap \{\phi_{xx}(t_0, x_0, y_0) < 0\}$ the maps H_n and C_n are real-valued and continuous by Lemma E.4 and Lemma E.5, respectively. In fact, on $\{\phi_x(t_0, x_0, y_0) > 0\} \cap \{\phi_{xx}(t_0, x_0, y_0) < 0\}$ both H_n and C_n are uniformly continuous in n because their coefficients converge locally uniformly by Assumption (A.E.8) and because $\sigma_n \geq \underline{\sigma} > 0$ by Assumption (A.E.9). Hence, we may pass to the limit in the operator:

$$F_{n_k}\big(t_k, x_k, y_k, \phi_t, D\phi, D^2\phi\big) \longrightarrow F\big(t_0, x_0, y_0, \phi_t(t_0, x_0, y_0), D\phi(t_0, x_0, y_0), D^2\phi(t_0, x_0, y_0)\big),$$

so that

$$F_*\big(t_0, x_0, y_0, \phi_t(t_0, x_0, y_0), D\phi(t_0, x_0, y_0), D^2\phi(t_0, x_0, y_0)\big) \leq 0.$$

Thus $\overline{J}$ is a viscosity subsolution of the limiting HJB equation.

Step 3: $\underline{J}$ is a viscosity supersolution of the limiting equation. Let $\psi \in C^{1,2}$ be such that $\underline{J} - \psi$ attains a strict local minimum at $(t_0, x_0, y_0) \in (0, T) \times \mathrm{int}(K)$ and

$$(\underline{J} - \psi)(t_0, x_0, y_0) = 0.$$

Applying again the half-relaxed-limits argument of Crandall, Ishii, and Lions 1992, Lemma 6.1 and Remark 6.3, we find integers $n_k \to \infty$ and points

$$(t_k, x_k, y_k) \to (t_0, x_0, y_0)$$

such that

$$J_{n_k}(t_k, x_k, y_k) \to \underline{J}(t_0, x_0, y_0),$$

and $J_{n_k} - \psi$ attains a local minimum at (t_k, x_k, y_k).

Since each J_{n_k} is a viscosity supersolution, we have

$$F_{n_k}\big(t_k, x_k, y_k, \psi_t(t_k, x_k, y_k), D\psi(t_k, x_k, y_k), D^2\psi(t_k, x_k, y_k)\big) \geq 0. \tag{E.11}$$

We first note that the above inequality (E.11) implies

$$H_{n_k}\big(t_k, x_k, y_k, D\psi(t_k, x_k, y_k), D^2\psi(t_k, x_k, y_k)\big) < \infty, \qquad C_{n_k}(t_k, D\psi(t_k, x_k, y_k)) < \infty,$$

since otherwise the left-hand side in (E.11) is $-\infty$. By Lemma E.4 and Lemma E.5 this fact implies $\psi_x(t_k, x_k, y_k) \geq 0$ and $\psi_{xx}(t_k, x_k, y_k) \leq 0$. Now, we may rewrite (E.11) as

$$\begin{aligned} &C_{n_k}(t_k, D\psi(t_k, x_k, y_k)) + H_{n_k}\big(t_k, x_k, y_k, D\psi(t_k, x_k, y_k), D^2\psi(t_k, x_k, y_k)\big) \\ &\qquad \leq -\psi_t(t_k, x_k, y_k) - r x_k\, \psi_x(t_k, x_k, y_k) - b_{n_k}(y_k)\, \psi_y(t_k, x_k, y_k) - \frac{a_{n_k}}{2}\psi_{yy}(t_k, x_k, y_k), \end{aligned}$$

whose right-hand side converges to

$$-\psi_t(t_0, x_0, y_0) - r x_0\, \psi_x(t_0, x_0, y_0) - b(t_0, y_0)\, \psi_y(t_0, x_0, y_0) - \frac{a}{2}\psi_{yy}(t_0, x_0, y_0)$$

by Assumption (A.E.8) and the fact that $\psi \in C^{1,2}$. From Lemma E.4 and Lemma E.5 together with Assumptions (A.E.8) and (A.E.9) the Hamiltonians are suprema of affine functions whose coefficients converge, and therefore

$$\begin{aligned}\liminf_{k\to\infty} C_{n_k}(t_k,\psi_x) &= \liminf_{k\to\infty}\sup_{c\in\mathbb{R}}\big(\,\mathrm{e}^{-rt}\,U(a_i,c) - c\,\psi_x(t_k,x_k,y_k)\big)\\ &\geq \sup_{c\in\mathbb{R}}\liminf_{k\to\infty}\big(\,\mathrm{e}^{-rt}\,U(a_i,c) - c\,\psi_x(t_k,x_k,y_k)\big) = C(t_0,D\psi(t_0,x_0,y_0)),\end{aligned}$$

and similarly

$$\liminf_{k\to\infty} H_{n_k}\big(t_k,x_k,y_k,D\psi,D^2\psi\big) \geq H\big(t_0,x_0,y_0,D\psi(t_0,x_0,y_0),D^2\psi(t_0,x_0,y_0)\big),$$

so that

$$\begin{aligned}&\liminf_{k\to\infty}\big(C_{n_k}(t_k,D\psi(t_k,x_k,y_k)) + H_{n_k}\big(t_k,x_k,y_k,D\psi(t_k,x_k,y_k),D^2\psi(t_k,x_k,y_k)\big)\big)\\ &\qquad\geq \liminf_{k\to\infty} C_{n_k}(t_k,D\psi(t_k,x_k,y_k)) + \liminf_{k\to\infty} H_{n_k}\big(t_k,x_k,y_k,D\psi(t_k,x_k,y_k),D^2\psi(t_k,x_k,y_k)\big)\\ &\qquad\geq C(t_0,D\psi(t_0,x_0,y_0)) + H\big(t_0,x_0,y_0,D\psi(t_0,x_0,y_0),D^2\psi(t_0,x_0,y_0)\big),\end{aligned}$$

which together with the above eventually implies

$$F\big(t_0,x_0,y_0,\psi_t(t_0,x_0,y_0),D\psi(t_0,x_0,y_0),D^2\psi(t_0,x_0,y_0)\big) \geq 0.$$

Hence $\underline{J}$ is a viscosity supersolution of the limiting HJB equation.

Step 4: comparison and identification of the limit. The goal of this step is to conclude $\overline{J} \leq \underline{J}$. The idea is to apply the comparison principle, which is proven in Lemma E.14 below, to the pair $(\overline{J},\underline{J})$. To do so we have to show that the assumptions of Lemma E.14 are satisfied.

(a) The first requirement is that $\overline{J}$ is an upper semicontinuous subsolution of $F_* = 0$. Upper semicontinuity is proven in Step 1, and the subsolution property is confirmed in Step 2. Recall from Corollary E.6 that the inequality is asserted only at jets with $\phi_x > 0$ and $\phi_{xx} < 0$; by (d) below this is harmless.

(b) The second requirement is that $\underline{J}$ is a lower semicontinuous supersolution of $F = 0$. Again, lower semicontinuity is proven in Step 1, and the supersolution property is confirmed in Step 3.

(c) Both, $\overline{J}$ and $\underline{J}$, must satisfy the growth bound of Lemma E.14. By assumption (A.E.10) this requirement is satisfied since $\frac{\mathrm{e}^{rt}\,|J_n|}{\varrho}$ is bounded on $[0,\infty)\times I_x\times\mathbb{R}$ uniformly in n and both half-relaxed limits inherit that bound. Assumption (iii) of Lemma E.14 is covered by Assumption (A.E.12).

(d) Finally, we need that $\underline{J}$ is uniformly strictly concave in wealth. This requirement is covered by Lemma E.10, which uses Lemma E.9, and Assumptions (A.E.10)–(A.E.11).

Hence Lemma E.14 applies and yields $\overline{J} \le \underline{J}$ on $[0,\infty) \times E$. Since always $\underline{J} \le \overline{J}$ by Step 1, we conclude

$$\overline{J} = \underline{J} =: J \qquad \text{on } [0,\infty) \times E,$$

so that J is both upper and lower semicontinuous, hence continuous, and is a viscosity solution of the limiting equation. Moreover $J = \underline{J}$ is uniformly strictly concave in wealth on compacts, and uniqueness of J in that class follows from Lemma E.14 applied to two solutions. ∎

E.5. Comparison lemma

By Assumption (A.E.8) the limiting operator is

$$F(t,x,y,q,p,X) = -q - rx\,p_x - b(t,y)\,p_y - \frac{a}{2}X_{yy} - C(t,p_x) - H(t,y,p_x,X_{xx}), \tag{E.12}$$

with $C(t,p_x) = \sup_{c\in\mathbb{R}}(\mathrm{e}^{-rt}\,U(a_i,c) - cp_x)$ and, since $d = 0$, $H = -\frac{(m(t,y)p_x)^2}{2\sigma X_{xx}}$ on $\{X_{xx} < 0\}$. We first prove that the limiting operator F is proper in the sense of condition (0.1) in Crandall, Ishii, and Lions (1992).

Lemma E.12. *Let Assumption (A.E.8) hold. Then the operator F defined in* (E.12) *is proper, i.e.,*

$$F(t,x,y,q,p,X) \le F(t,x,y,q,p,Y) \qquad \forall\, Y \le X.$$

Proof. Note that $X - Y$ is positive semi-definite by definition of $Y \le X$, and therefore

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}X_{xx}-Y_{xx} & X_{xy}-Y_{xy}\\ X_{xy}-Y_{xy} & X_{yy}-Y_{yy}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = X_{xx} - Y_{xx} \ge 0,$$

and similarly

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}X_{xx}-Y_{xx} & X_{xy}-Y_{xy}\\ X_{xy}-Y_{xy} & X_{yy}-Y_{yy}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = X_{yy} - Y_{yy} \ge 0.$$

Consequently, it holds

$$hm(t,y) + \frac{1}{2}h^2\sigma X_{xx} \ge hm(t,y) + \frac{1}{2}h^2\sigma Y_{xx} \qquad \forall h \in \mathbb{R},$$

so that $H(t,x,y,q,p,X) \ge H(t,x,y,q,p,Y)$. Hence, since $a \ge 0$ by (A.E.8), we eventually obtain from

$$F(t,x,y,q,p,X) - F(t,x,y,q,p,Y) = -\frac{a}{2}(X_{yy} - Y_{yy}) - (H(t,x,y,q,p,X) - H(t,x,y,q,p,Y))$$

that $F(t,x,y,q,p,X) - F(t,x,y,q,p,Y) \le 0$, and this concludes the proof. ∎

Note that Lemma E.12 only requires $Y \leq X$, but not also $r \leq s$ as in condition (0.1) of Crandall, Ishii, and Lions (1992). The reason is that the limiting operator F only depends on first and second order partial derivatives, but not on the zeroth order.

Next, we prove that the limiting operator F satisfies condition (3.14) in Crandall, Ishii, and Lions (1992).

Lemma E.13. *Let Assumptions (A.E.8) and (A.E.9) hold, and let u be an upper semicontinuous subsolution of $F_* = 0$, and v a lower semicontinuous supersolution of $F = 0$ on $(0, \infty) \times E$, and assume that the following assertions are satisfied:*

(i) for every compact interval $I_x \subset (0, \infty)$, it holds $\sup_{[0,\infty)\times I_x \times \mathbb{R}} \frac{\mathrm{e}^{rt}(|u|+|v|)}{\varrho(y)} < \infty$, where $\varrho(y) = 1 + |y|^\kappa$ for some $\kappa \in [0, 2)$ as in Assumption (A.E.10);

(ii) for every $T > 0$ and every compact $K \subset (0, \infty) \times \mathbb{R}$ there are $c_{T,K} > 0$ and $\lambda_{T,K} > 0$ such that $x \mapsto v(t, x, y) - c_{T,K}x$ is non-decreasing and $x \mapsto v(t, x, y) + \frac{\lambda_{T,K}}{2}x^2$ is concave on the x-sections of K, for all $t \in [0, T]$;

(iii) $\lim_{x\downarrow 0} \sup_{(t,y)} \frac{\mathrm{e}^{rt}(u-v)^+}{\varrho(y)} = \lim_{x\to\infty} \sup_{(t,y)} \frac{\mathrm{e}^{rt}(u-v)^+}{\varrho(y)} = 0$, with $\varrho(y) = 1 + |y|^\kappa$ for some $\kappa \in [0, 2)$ as in (i).

Let $K \subseteq E$ be a compact subset of E. If $\alpha > 0$ and $(t, z_1), (t, z_2) \in [0, T] \times \mathrm{int}(K)$ are such that

$$(\alpha(z_1 - z_2), X) \in \overline{\mathcal{J}}^{2,+}u(t, z_1), \qquad \textit{and} \qquad (\alpha(z_1 - z_2), Y) \in \overline{\mathcal{J}}^{2,-}v(t, z_2),$$

with $X, Y \in \mathbb{S}^2$ satisfying

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{E.13}$$

then there exists a constant $c_{T,K,K'} \in (0, \infty)$ such that

$$F(t, z_2, \alpha(z_1 - z_2), Y) - F(t, z_1, \alpha(z_1 - z_2), X) \leq c_{T,K,K'}(\alpha\|z_1 - z_2\|^2 + \|z_1 - z_2\|).$$

Proof. Let us first note that the following facts hold:

(a) Since the function v satisfies assertion (ii), which is the analogue of Assumption (A.E.11), we can adopt the proof of Lemma E.8 to show that $Y_{xx} \leq -\lambda_{T,K}$ and $p_x \geq c_{T,K}$.

(b) By the comment after condition (3.10) in Crandall, Ishii, and Lions (1992), it follows from (E.13) that $X \leq Y$. Hence, in the same spirit as in the proof of Lemma E.12 we find that $Y_{xx} - X_{xx} \geq 0$, and thus by assertion (a) it holds $X_{xx} \leq Y_{xx} \leq -\lambda_{T,K} < 0$.

(c) Since assumptions (i) and (ii) are the analogues of Assumptions (A.E.10) and (A.E.11), we may adopt the proof of Lemma E.9 to show that for every $T > 0$ and every compact subset $K' \subset \operatorname{int}(K)$ there exists $L_{T,K,K'} < \infty$ with

$$|v(t,x,y) - v(t,x',y)| \le L_{T,K,K'}|x - x'| \qquad \forall\,(t,x,y),\,(t,x',y) \in [0,T] \times K'.$$

Now, by the definition of the subjet it holds that

$$v(t,z) \ge v(t,z_2) + \alpha(z_1 - z_2)^\mathsf{T}(z - z_2) + \frac{1}{2}(z - z_2)^\mathsf{T} Y(z - z_2) + o(\|z - z_2\|^2) \quad z \to z_2,$$

and by taking $z = (x_2 + h, y_2)$ for h small this inequality becomes

$$v(t, x_2 + h, y_2) \ge v(t, x_2, y_2) + \alpha(x_1 - x_2)h + \tfrac{1}{2}h^2 Y_{xx} + o(|h|^2),$$

and so together with the fact that v is Lipschitz in x we eventually have

$$\alpha(x_1 - x_2)h + \tfrac{1}{2}h^2 Y_{xx} + o(|h|^2) \le v(t, x_2 + h, y_2) - v(t, x_2, y_2) \le L_{T,K,K'}|h|.$$

Now, if $x_1 - x_2 \le 0$ take $h < 0$, and if $x_1 - x_2 \ge 0$ take $h > 0$, so that in both cases we have

$$\alpha|x_1 - x_2| \cdot |h| + \tfrac{1}{2}h^2 Y_{xx} + o(|h|^2) \le L_{T,K,K'}|h|.$$

Hence, dividing first by $|h|$, and then letting either $h \nearrow 0$ if $h < 0$ or $h \searrow 0$ if $h > 0$, we eventually obtain $\alpha|x_1 - x_2| \le L_{T,K,K'}$, and therefore $|p_x| \le 2L_{T,K,K'}$.

Now, let us analyze the difference of the operators. In particular, we have

$$\begin{aligned}
&F(t, z_2, \alpha(z_1 - z_2), Y) - F(t, z_1, \alpha(z_1 - z_2), X) \\
&\qquad = r(x_1 - x_2)p_x + (b(t, y_1) - b(t, y_2))p_y + \frac{a}{2}(X_{yy} - Y_{yy}) + H(t, z_1, p, X) - H(t, z_2, p, Y),
\end{aligned}$$

and we analyze this difference term by term.

(1) To analyze the term $r(x_1 - x_2)p_x + (b(t,y_1) - b(t,y_2))p_y$, let us first note that by assumption $p = (p_x, p_y) = \alpha(z_1 - z_2)$, and thus $p_x = \alpha(x_1 - x_2)$ and $p_y = \alpha(y_1 - y_2)$. Therefore, it holds

$$\begin{aligned}
r(x_1 - x_2)p_x + (b(t, y_1) - b(t, y_2))p_y &= r\alpha(x_1 - x_2)^2 + (b(t, y_1) - b(t, y_2))\alpha(y_1 - y_2) \\
&\le (r + \operatorname{Lip}(b))\alpha\|z_1 - z_2\|^2,
\end{aligned}$$

where we use from Assumption (A.E.8) that b is locally Lipschitz-continuous, uniformly in t on compacts.

(2) To analyze the difference $\frac{a}{2}(X_{yy} - Y_{yy})$ we note as in the proof of Lemma E.12 that $X \le Y$ implies $Y_{yy} - X_{yy} \ge 0$, which together with $a \ge 0$ by Assumption (A.E.8) implies the upper bound $\frac{a}{2}(X_{yy} - Y_{yy}) \le 0$.

(3) Note that the consumption Hamiltonian $C(t, p_x)$ cancels exactly because the doubling is in space only, which means that t and p_x are common to the two sides. In particular no regularity of $t \mapsto C(t, \cdot)$ is needed.

(4) Finally, for the portfolio Hamiltonian $H(\cdot)$ compare the suprema for each h. For this purpose, we first note that by Assumption (A.E.8) the function m is locally Lipschitz-continuous in y, uniformly in t, so that

$$m(t, y_1) - m(t, y_2) \leq |m(t, y_1) - m(t, y_2)| \leq \mathrm{Lip}(m)|y_1 - y_2|.$$

Then, since by assertion (b) it holds $X_{xx} \leq Y_{xx}$, we find that

$$h\, m(t, y_1)p_x + \frac{h^2}{2}\sigma X_{xx} \leq h\, m(t, y_2)p_x + \frac{h^2}{2}\sigma Y_{xx} + |h| \,\mathrm{Lip}(m)|y_1 - y_2| \cdot |p_x| \qquad \forall\, h \in \mathbb{R}.$$

From assertion (c) we further know that

$$|h| \,\mathrm{Lip}(m)|y_1 - y_2| \cdot |p_x| = |h| \,\mathrm{Lip}(m)|y_1 - y_2| \cdot \alpha|x_1 - x_2| \leq |h| \,\mathrm{Lip}(m)|y_1 - y_2| 2L_{T,K,K'}.$$

However, since $X_{xx} \leq Y_{xx} \leq -\lambda_{T,K} < 0$ by assertion (a), the portfolio Hamiltonian $H(\cdot)$ on the left-hand side above satisfies $H = -\frac{(m(t,y_1)p_x)^2}{2\sigma X_{xx}}$. More precisely, on the left-hand side above, the supremum is attained at $h = -\frac{m(t,y_1)p_x}{\sigma X_{xx}}$, which implies

$$|h| = \frac{|m(t, y_1)| \cdot |p_x|}{\sigma |X_{xx}|} \leq \sup_{(x,y)\in K} \frac{\mathrm{Lip}(m)|y| 2L_{T,K,K'}}{\sigma \lambda_{T,K,K'}} =: \tilde{c}_{T,K,K'}$$

Therefore, we eventually find that

$$H(t, z_1, p, X) \leq H(t, z_2, p, Y) + \tilde{c}_{T,K,K'} \,\mathrm{Lip}(m)|y_1 - y_2| 2L_{T,K,K'}.$$

By combining assertions (1)–(4) we find that

$$\begin{aligned}
&F(t, z_2, \alpha(z_1 - z_2), Y) - F(t, z_1, \alpha(z_1 - z_2), X) \\
&\qquad \leq (r + \mathrm{Lip}(b))\alpha\|z_1 - z_2\|^2 + 2\tilde{c}_{T,K,K'} L_{T,K,K'} \,\mathrm{Lip}(m)\|z_1 - z_2\| \\
&\qquad \leq (r + \mathrm{Lip}(b) + 2\tilde{c}_{T,K,K'} L_{T,K,K'} \,\mathrm{Lip}(m))(\alpha\|z_1 - z_2\|^2 + \|z_1 - z_2\|),
\end{aligned}$$

and setting $c_{T,K,K'} := r + \mathrm{Lip}(b) + 2\tilde{c}_{T,K,K'} L_{T,K,K'} \,\mathrm{Lip}(m)$ is enough to conclude the proof. ■

With Lemma E.12 and Lemma E.13 in hand we can now state and prove the comparison result needed for the proof of Theorem E.7.

Lemma E.14. *Let Assumptions (A.E.8) and (A.E.9) hold, and let u be an upper semicontinuous subsolution of $F_* = 0$, and v a lower semicontinuous supersolution of $F = 0$ on*

$(0,\infty) \times E$, *and assume the following assertions hold true:*

(i) for every compact interval $I_x \subset (0,\infty)$, *it holds* $\sup_{[0,\infty)\times I_x\times\mathbb{R}} \frac{\mathrm{e}^{rt}(|u|+|v|)}{\varrho(y)} < \infty$, *where* $\varrho(y) = 1 + |y|^\kappa$ *for some* $\kappa \in [0,2)$ *as in Assumption (A.E.10);*

(ii) for every $T > 0$ *and every compact* $K \subset (0,\infty) \times \mathbb{R}$ *there are* $c_{T,K} > 0$ *and* $\lambda_{T,K} > 0$ *such that* $x \mapsto v(t,x,y) - c_{T,K}x$ *is non-decreasing and* $x \mapsto v(t,x,y) + \frac{\lambda_{T,K}}{2}x^2$ *is concave on the* x*-sections of* K*, for all* $t \in [0,T]$*;*

(iii) $\lim_{x\downarrow 0}\sup_{(t,y)} \frac{\mathrm{e}^{rt}(u-v)^+}{\varrho(y)} = \lim_{x\to\infty}\sup_{(t,y)} \frac{\mathrm{e}^{rt}(u-v)^+}{\varrho(y)} = 0$, *with* $\varrho(y) = 1 + |y|^\kappa$ *for some* $\kappa \in [0,2)$ *as in (i).*

Then $u \le v$ *on* $[0,\infty) \times E$*.*

Proof. Fix $\varepsilon > 0$ and $\delta > 0$. By assumption (iii) there are $0 < x_- < x_+ < \infty$, such that

$$u(t,x,y) - v(t,x,y) \le \varepsilon\,\mathrm{e}^{-rt}\,\varrho(y) \qquad \text{for all } x \le x_- \text{ or } x \ge x_+. \tag{E.14}$$

Moreover, by letting $I_x \coloneqq [x_-, x_+]$ and

$$\iota \coloneqq \sup_{(t,x,y)\in[0,\infty)\times I_x\times\mathbb{R}} \frac{\mathrm{e}^{rt}(|u(t,x,y)| + |v(t,x,y)|)}{\varrho(y)} < \infty,$$

it holds that $\iota < \infty$ by assumption (i), and further

$$\iota\,\mathrm{e}^{-rt}\,\varrho(y) \ge (|u(t,x,y)| + |v(t,x,y)|) \ge u(t,x,y) - v(t,x,y). \tag{E.15}$$

Now, let us fix $M_0 > \frac{a}{r}$ and set

$$\chi(t,y) \coloneqq \mathrm{e}^{-rt}(M_0 + y^2), \qquad v_\delta \coloneqq v + \delta\chi.$$

Since χ does not depend on x, we have $\chi_x = \chi_{xx} = \chi_{xy} = 0$, so C and H in (E.12) are unchanged by the perturbation, and v_δ still satisfies (ii). Moreover, for every subjet of v at the point (t,x,y), it holds that

$$F(\cdot, \psi_t + \delta\chi_t, D\psi + \delta D\chi, D^2\psi + \delta D^2\chi) = F(\cdot, \psi_t, D\psi, D^2\psi) + \delta\frac{r(M_0 + y^2) - 2b(t,y)y - a}{\mathrm{e}^{rt}},$$

and by Assumption (A.E.8) it also holds that $2b(t,y)y \le ry^2$, so the numerator in the second term of the right-hand side in the above equation is at least $rM_0 - a > 0$. The strict inequality here holds by our initial assumption $M_0 > \frac{a}{r}$. Hence v_δ is again a supersolution, and $v_\delta \ge v$. Since $\varrho(y) = 1 + |y|^\kappa$ with $\kappa < 2$, there is $R = R_\delta$ such that $\delta y^2 > 2\iota\varrho(y)$ for all $|y| \ge R$. Indeed, dividing both sides by y^2 yields

$$\delta > 2\iota\frac{\varrho(y)}{y^2} = 2\iota(y^{-2} + |y|^{\kappa-2}) \to 0 \qquad \text{as } y \to \infty,$$

since $\kappa < 2$ by assumption (iii). By choosing R large enough, it holds that $\delta > 2\iota(y^{-2}+y^{\kappa-2})$ for all $|y| \geq R$. Hence, together with (E.15) it holds that

$$u - v_\delta \leq 2\iota\, \mathrm{e}^{-rt}\, \varrho(y) - \delta\, \mathrm{e}^{-rt}(M_0 + y^2) < 0 \qquad \text{on } [0,\infty) \times I_x \times \{|y| \geq R\}. \tag{E.16}$$

We already know from (E.15) that $u - v \leq 2\iota\, \mathrm{e}^{-rt}\, \varrho(y)$ on $[0,\infty) \times I_x \times \mathbb{R}$. Now, choose $T > 0$ with $2\iota\, \mathrm{e}^{-rT}\, \varrho(R) \leq \varepsilon$, so that

$$u - v_\delta \leq u - v \leq \varepsilon \qquad \text{on } \{T\} \times I_x \times [-R, R]. \tag{E.17}$$

Then, consider the set

$$Q \coloneqq (0,T) \times (x_-, x_+) \times (-R, R).$$

Since the operator (E.12) contains no zeroth-order term, $v_\delta + \varepsilon'$ is again a supersolution for every constant ε'. For our purpose, set $\varepsilon' \coloneqq \varepsilon\varrho(R)$. By combining (E.14), (E.16) and (E.17), we thus obtain $u \leq v_\delta + \varepsilon'$ on the whole parabolic boundary of Q (the two lateral faces in x, the two lateral faces in y, and the terminal face $t = T$). On $\overline{Q}$ the functions u and v_δ are bounded, v_δ satisfies (ii) with a single constant $\lambda \coloneqq \lambda_{T,\overline{Q}}$, and by the same argumentation as in the proof of Lemma E.9 it is Lipschitz-continuous in x, uniformly in (t,y), with a Lipschitz-constant $L_{T,\overline{Q}}$. In particular, if $(p, X) \in \mathcal{J}^{2,-}v_\delta(t,x,y)$, then it follows by the same argumentation as in Lemma E.8 that $X_{xx} \leq -\lambda_{T,\overline{Q}}$. The reason is that v_δ satisfies assumption (ii) which is the analogue of (A.E.11) for the limiting operator. On the other hand, by the subjet inequality evaluated at (t, x', y) it holds that

$$v_\delta(t,x',y) - v_\delta(t,x,y) \geq p_x(x'-x) + \frac{1}{2}(x'-x)^2 X_{xx} + o(|x'-x|^2),$$

so that letting $x' = x + h$ yields

$$v_\delta(t,x+h,y) - v_\delta(t,x,y) \geq p_x h + \frac{1}{2}h^2 X_{xx} + o(|h|^2).$$

However, we may note that by assumption (ii) there is $c_{T,\bar{Q}} > 0$ so that

$$v_\delta(t,x+h,y) - v_\delta(t,x,y) = v(t,x+h,y) - v(t,x,y) \leq c_{T,\bar{Q}}h \qquad \forall\, h < 0.$$

Consequently, we find that $p_x h + \frac{1}{2}h^2 X_{xx} + o(|h|^2) \leq c_{T,\bar{Q}}h$ for all $h < 0$, so that dividing by h and letting $h \nearrow 0$ yields $p_x \geq c_{T,\bar{Q}}$. As a consequence, the operator $F(\cdot)$ is only needed on a set $\{X_{xx} \leq -\lambda_{T,\bar{Q}}\} \cap \{p_x \geq c_{T,\bar{Q}}\}$ on which it is continuous. The assertion follows by the same argumentation as in the proofs of Lemma E.4, Lemma E.5, and Corollary E.6. The fact that F is not globally continuous is therefore immaterial, and the distinction between F and

F_* disappears at the doubled points. Together with Lemma E.12 and Lemma E.13, which show that F is proper and satisfies condition (3.14) in Crandall, Ishii, and Lions (1992), we may thus adapt the proof of Theorem 8.2 in Crandall, Ishii, and Lions (1992) to conclude that

$$u \le v_\delta + \varepsilon\varrho(R) \qquad \text{on } \overline{Q}.$$

Together with (E.14), (E.16) and the bound $u - v \le 2\iota\, \mathrm{e}^{-rt}\, \varrho(y)$ for $t \ge T$, the same inequality holds on all of $[0, \infty) \times E$.

Letting $\delta \searrow 0$ gives $u \le v + \varepsilon\varrho(R)$ pointwise. Here it is worth noting that R was chosen before δ. Then, by letting $\varepsilon \searrow 0$ we obtain $u \le v$, and this concludes the proof. ■

E.6. Convergence of the first order partial derivatives

We use the following standard definitions. A vector $x^* \in \mathbb{R}^n$ is a *supergradient* of the function $f\colon C \subseteq \mathbb{R}^n \to \mathbb{R}$ at x, if for all $y \in C$ it holds $f(y) \le f(x) + \langle x^*, y - x\rangle$. The *superdifferential* of f at x is then the set of all supergradients of f at x, and we write

$$\partial^+ f(x) \coloneqq \big\{x^* \in \mathbb{R}^n \colon f(y) \le f(x) + \langle x^*, y - x\rangle \,\forall y \in C\big\}.$$

We have the following result.

Lemma E.15. *Let Assumption (A.E.11) hold on $[0, T] \times K$ and let $J_n \to J$ uniformly on $[0, T] \times K$, with $J(t, \cdot, y)$ continuously differentiable on the x-sections of* $\operatorname{int}(K)$*, uniformly in (t, y). Then for every compact $K' \subset \operatorname{int}(K)$,*

$$\sup_{[0,T]\times K'} \big|\partial_x J_n(t, x, y) - \partial_x J(t, x, y)\big| \to 0, \qquad \textit{as } n \to \infty.$$

where $\partial_x J_n$ may be read as any selection of the (nonempty) x-superdifferential of the semi-concave function $x \mapsto J_n(t, x, y)$.

Proof. Write $\lambda = \lambda_{T,K}$ and $g_n(t, \cdot, y) = J_n(t, \cdot, y) + \frac{\lambda}{2}x^2$, which is concave by Assumption (A.E.11). Concavity of g_n implies for any $\alpha \in [0, 1]$ and any $x, x' \in K$ that

$$\begin{aligned} &J_n(t, \alpha x + (1-\alpha)x', y) + \tfrac{\lambda}{2}(\alpha x + (1-\alpha)x')^2 \\ &\qquad \ge \alpha J_n(t, x, y) + (1-\alpha)J_n(t, x', y) + \tfrac{\lambda}{2}(\alpha x^2 + (1-\alpha)(x')^2), \end{aligned}$$

which together with convexity of $x \mapsto x^2$ implies that $J_n(t, \cdot, y)$ is concave. Hence, by Theorem 23.4 in Rockafellar (1970), for a point $(x, y) \in K' \subset \operatorname{int}(K)$ the x-superdifferential of $J_n(t, \cdot, y)$ at x is non-empty, i.e., there exists a supergradient $p_n \in \mathbb{R}$ of $J_n(t, \cdot, y)$ at x and therefore satisfies

$$J_n(t, \tilde{x}, y) \le J_n(t, x, y) + p_n(\tilde{x} - x) \qquad \forall\, (\tilde{x}, y) \in K.$$

Now, by taking $\tilde{x} = x \pm h$ for some $h > 0$ small, we find that

$$\frac{J_n(t,x+h,y) - J_n(t,x,y)}{h} \leq p_n \leq \frac{J_n(t,x,y) - J_n(t,x-h,y)}{h}. \tag{E.18}$$

Since $J_n \to J$ locally uniformly on $[0,\infty) \times E$, it follows from (E.18) that

$$\frac{J(t,x+h,y) - J(t,x,y)}{h} \leq \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n \leq \frac{J(t,x,y) - J(t,x-h,y)}{h}.$$

Moreover, since $J(t,\cdot,y)$ is continuously differentiable by assumption, letting $h \searrow 0$ then yields

$$\lim_{h\searrow 0} \frac{J(t,x+h,y) - J(t,x,y)}{h} = J_x(t,x,y) = \lim_{h\searrow 0} \frac{J(t,x,y) - J(t,x-h,y)}{h}.$$

Hence, the two limits agree. Uniform convergence then follows from Theorem 25.7 in Rockafellar (1970). This concludes the proof. ■

E.7. Convergence of the second order partial derivatives

(A.E.13) *For every $T > 0$ and every compact subset $K \subseteq E$ the functions $(J_n)_{n\in\mathbb{N}}$ belong to $C^{1,2}([0,T]\times K)$, and the families $(J_{t,n})_{n\in\mathbb{N}}$, $(DJ_n)_{n\in\mathbb{N}}$, and $(D^2J_n)_{n\in\mathbb{N}}$ are uniformly bounded and equicontinuous on $[0,T]\times K$.*

Proposition E.16. *Let the assumptions of Theorem E.7 hold, and assume that (A.E.4) and (A.E.13) hold as well. Then the limit J in Theorem E.7 is in $C^{1,2}([0,\infty)\times E)$, is a classical solution to the limiting HJB equation, and*

$$J_{n,t} \to J_t, \qquad DJ_n \to DJ, \qquad D^2J_n \to D^2J \qquad \text{locally uniformly on } [0,\infty)\times E.$$

In particular, $J_{n,x} \to J_x$, $J_{n,y} \to J_y$, $J_{n,xx} \to J_{xx}$, $J_{n,xy} \to J_{xy}$, and $J_{n,yy} \to J_{yy}$ locally uniformly, and $J_{xx} \leq -\lambda_{T,K} < 0$ on $[0,T]\times K$.

Proof. The idea is to apply the Arzelà-Ascoli Theorem, which is possible due to Assumption (A.E.13). In particular, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that each member of the family of functions $(J_n, J_{n,t}, DJ_n, D^2J_n)_{n\in\mathbb{N}}$ converges uniformly to a continuous function in $C^{1,2}([0,T]\times K)$. Since every sequence has a further subsequence with the same limit, the whole sequence converges. Now, uniform convergence of a sequence of functions together with uniform convergence of their partial derivatives implies that the limit is differentiable with the corresponding derivatives. In other words, if $(J_{n_k}, J_{n_k,t}, DJ_{n_k}, D^2J_{n_k}) \to (v, v_0, v_1, v_2)$, then it must hold that $(v, v_0, v_1, v_2) = (v, v_t, Dv, D^2v)$. From Theorem E.7 we know that $J_n \to J$ locally uniformly, thus uniformly on $[0,T]\times K$, which implies that $v = J$ and thus

$J \in C^{1,2}([0,T] \times K)$. Consequently, we have $(v, v_t, Dv, D^2 v) = (J, J_t, DJ, D^2 J)$, which is enough to prove local uniform convergence.

It remains to prove that J is a classical solution to the limiting HJB equation. We obtain this fact by using Assumption (A.E.11) which states that $x \mapsto J_n(t,x,y) + \frac{\lambda_{T,K}}{2} x^2$ is concave for some $\lambda_{T,K} > 0$ on every x-section of K. Hence, we find that $J_{n,xx}(t,x,y) + \lambda_{T,K} \leq 0$, and thus $J_{n,xx}(t,x,y) \leq -\lambda_{T,K} < 0$. On the other hand, $J_n(t,x,y)$ is a classical solution, and so by Lemma E.5 we must have $J_{n,x}(t,x,y) \geq 0$ for the consumption Hamiltonian to be finite. Together with the fact that $(U')^{-1}$ is defined on $(0,\infty)$ the case $J_{n,x}(t,x,y) = 0$ is excluded, which implies $J_{n,x}(t,x,y) > 0$. Taking the pointwise limit yields $J_{xx} < 0$ and $J_x > 0$, and by Corollary E.6 the function $F(t,x,y,J_t,DJ,D^2J)$ is continuous on $\{J_{xx} < 0\} \cap \{J_x > 0\}$ so that $F_n \to F$. This concludes the proof. ■

Corollary E.17. *Let the assumptions from Proposition E.16 hold. Then the feedback maps defined in* (E.3) *and* (E.4) *converge locally uniformly on* $[0,\infty) \times E$*, that is*

$$c_n^* \to c^* = \big(U'(a_i, \cdot)\big)^{-1}(\mathrm{e}^{rt}\, J_x), \qquad h_n^* \to h^* = -\frac{m(t,y)J_x + dJ_{xy}}{\sigma J_{xx}} = -\frac{m(t,y)J_x}{\sigma J_{xx}},$$

and in particular

$$\frac{J_{n,x}}{J_{n,xx}} \to \frac{J_x}{J_{xx}}, \qquad \frac{J_{n,xy}}{J_{n,xx}} \to \frac{J_{xy}}{J_{xx}} \qquad \textit{locally uniformly on } [0,\infty) \times E.$$

Proof. By Proposition E.16 all of $J_{n,xx} \to J_{xx}$, $J_{n,xy} \to J_{xy}$, and $J_{n,yy} \to J_{yy}$ converge locally uniformly on $[0,\infty) \times E$, and for a compact subset $K \subseteq E$ and a finite time $T > 0$ we have $|J_{n,xx}| \geq \lambda_{T,K} > 0$, which implies that $\frac{J_{n,x}}{J_{n,xx}} \to \frac{J_x}{J_{xx}}$ and $\frac{J_{n,xy}}{J_{n,xx}} \to \frac{J_{xy}}{J_{xx}}$ locally uniformly on $[0,\infty) \times E$. On the other hand, it holds by (A.E.8) that $m_n \to m$, $d_n \to 0$, and $\sigma_n \to \sigma$ which imply $h_n^* \to h^*$. For $c_n^* \to c^*$ we use (A.E.1) and (A.E.4), which imply that $(U')^{-1}$ is continuous on $(0,\infty)$, together with $\mathrm{e}^{rt}\, J_{n,x} \to \mathrm{e}^{rt}\, J_x$ locally uniformly with $\mathrm{e}^{rt}\, J_x > 0$. ■

E.8. The solution to the limiting HJB equation

We now derive the solution to the limiting HJB equation, if such a limit exists.

Lemma E.18. *The HJB equation* (E.2) *in Proposition E.1 satisfies Assumptions (A.E.8) and (A.E.9) of Theorem E.7.*

Proof. Let us first prove that Assumption (A.E.8) of Theorem E.7 is satisfied. For this purpose we prove the three sub-conditions given by equations (E.8), (E.9), and (E.10). To prove equation (E.8), recall that $b_n(y) = \big(\bar\beta^n(a_i) - \alpha_\Lambda^n\big)y$, where

$$\bar\beta^n(a_i) := \Big(\boldsymbol{\Gamma}^{a_i,n} - \begin{pmatrix}\boldsymbol{\Gamma}^n & 0\end{pmatrix}\Big)\, \boldsymbol{Q}^{a_i,n}\underline{b}(a_i).$$

The vector $\underline{b}(a_i) \in \mathbb{R}^{3\times 1}$ is exogenous and comes from the measure change (see Section D.3). From Lemma C.6 we know that $\lim_{n\to\infty}(\Gamma_1^{a_i,n} - \Gamma_1^n) = 0$ and $\lim_{n\to\infty}(\Gamma_2^{a_i,n} - \Gamma_2^n) = 0$, which together with Lemma C.12 yields $\lim_{n\to\infty} \bar{\beta}^n(a_i) = 0$. On the other hand, it holds that $\lim_{n\to\infty} \alpha_\Lambda^n = \alpha_\Lambda$ (see Corollary C.3) with

$$\alpha_\Lambda \coloneqq \frac{\alpha_D^2 \gamma}{\sigma_D^2} + \frac{\alpha_G \sigma_\xi^2 + \mu^2 \alpha_G^2 \gamma}{\mu^2 \sigma_G^2 + \sigma_\xi^2}.$$

Therefore, by letting $b(y) = -\alpha_\Lambda y$, it holds that $\sup_{y\in I_y} |b_n(y) - b(y)| \to 0$ for every compact interval $I_y \subset \mathbb{R}$, and thus equation (E.8) is satisfied. Since b is linear in y it is Lipschitz continuous in y, and we further have $2b(y)y = -2\alpha_\Lambda y^2 \leq 0 \leq ry^2$ since $\alpha_\Lambda \geq 0$ by Corollary C.3.

To prove equation (E.9), recall that $m_n(y) = \tilde{\rho}^n + \big(s_G r + \tilde{\beta}^n(a_i)\big)y$, where

$$\tilde{\rho}^n = -\frac{r\rho}{n}, \qquad \tilde{\beta}^n(a_i) = s_G \alpha_\Lambda^n + \begin{pmatrix} s_D + s_G\Gamma_1^n & s_G\Gamma_2^n & 0 \end{pmatrix} \boldsymbol{Q}^{a_i,n} \underline{b}(a_i).$$

Again, the vector $\underline{b}(a_i) \in \mathbb{R}^{3\times 1}$ comes from the measure change (see Section D.3). Hence, we may define

$$\tilde{\beta}(a_i) \coloneqq s_G \alpha_\Lambda + \begin{pmatrix} s_D + s_G\Gamma_1 & s_G\Gamma_2 & 0 \end{pmatrix} \boldsymbol{Q} \underline{b}(a_i),$$

where $\Gamma_1 = \lim_{n\to\infty} \Gamma_1^n$, $\Gamma_2 = \lim_{n\to\infty} \Gamma_2^n$, and $\boldsymbol{Q} = \lim_{n\to\infty} \boldsymbol{Q}^{a_i,n}$ (see Lemma C.12). As a result, by letting $m(y) = \beta(a_i)y$, it holds that $\sup_{(t,y)\in[0,T]\times I_y} |m_n(t,y) - m(y)| \to 0$ for every $T > 0$ and every compact interval $I_y \subset \mathbb{R}$. This proves equation (E.9). Since m is linear in y it is also Lipschitz continuous in y.

Finally, to prove equation (E.10), it suffices to note that

$$\eta_{VV} = \lim_{n\to\infty} \eta_{VV}^{a_i,n} = \sigma_D^2 (s_D + s_G\Gamma_1)^2 + (\mu^2\sigma_G^2 + \sigma_\xi^2) s_G^2 (\Gamma_2)^2 > 0,$$

as well as $\lim_{n\to\infty} \eta_{\Lambda\Lambda}^{a_i,n} = 0$, and $\lim_{n\to\infty} \eta_{V\Lambda}^{a_i,n} = 0$. All three limits follow by definition (see (E.1)) together with Lemma C.12. Hence, equation (E.10) is satisfied with $a = 0$, $d = 0$, and $\sigma = \eta_{VV}$.

Next, we prove that Assumption (A.E.9) of Theorem E.7 is satisfied. For this purpose, it is enough to note that

$$\eta_{VV}^{a_i,n} = \sigma_D^2 (s_D + s_G\Gamma_1^n)^2 + (\mu_n^2\sigma_G^2 + \sigma_{\xi,n}^2) s_G^2 (\Gamma_2^n)^2,$$

which follows from the definition $\eta_{VV}^{a_i,n}$ in (E.1) together with Lemma C.12. From Lemma C.1

we further know that

$$\Gamma_1^n = \frac{\alpha_D \gamma^{a_i,n}}{\sigma_D^2}, \qquad \Gamma_2^n = \frac{(\mu_n - w(a_i)\overline{\nu}_{a_i,n})(\sigma_G^2 - \alpha_G \gamma^{a_i,n})}{\tilde{\mu}_{a_i,n}^2 \sigma_G^2 + \tilde{\sigma}_{a_i,n}^2}.$$

Since $s_D = \frac{1}{r+\alpha_D}$ and $s_G = \frac{s_D \alpha_D}{r+\alpha_G}$ it is enough to note that $\gamma^{a_i,n} \geq 0$ for all $n \in \mathbb{N}$, as well as $\alpha_G > 0$, which implies

$$\eta_{VV}^{a_i,n} = s_D^2 \sigma_D^2 \left(1 + \frac{\alpha_D^2}{\sigma_D^2 (r+\alpha_G)} \gamma^{a_i,n}\right)^2 + (\mu_n^2 \sigma_G^2 + \sigma_{\xi,n}^2) s_G^2 (\Gamma_2^n)^2 \geq s_D^2 \sigma_D^2 > 0 \qquad \forall n \in \mathbb{N}.$$

This proves that Assumption (A.E.9) is satisfied. ■

Lemma E.19. *The coefficients $\eta_{V\Lambda}^{a_i,n}$ and $\eta_{\Lambda\Lambda}^{a_i,n}$ defined in* (E.1) *satisfy* $\lim_{n\to\infty} \frac{\eta_{V\Lambda}^{a_i,n}}{\sqrt{\eta_{\Lambda\Lambda}^{a_i,n}}} = 0$.

Proof. From the definition of $\eta_{\Lambda\Lambda}^{a_i,n}$ (see (E.1)) we know that

$$\begin{aligned}
\eta_{\Lambda\Lambda}^{a_i,n} &= \left(\mathbf{\Gamma}^{a_i,n} - \begin{pmatrix}\mathbf{\Gamma}^n & 0\end{pmatrix}\right) \boldsymbol{Q}^{a_i,n} \boldsymbol{Q}^{a_i,n} \left(\mathbf{\Gamma}^{a_i,n} - \begin{pmatrix}\mathbf{\Gamma}^n & 0\end{pmatrix}\right)^\mathsf{T} \\
&= \sigma_D^2 (\Gamma_1^{a_i,n} - \Gamma_1^n)^2 + (\mu_n^2 \sigma_G^2 + \sigma_{\xi,n}^2)(\Gamma_2^{a_i,n} - \Gamma_2^n)^2 + (\overline{\nu}_{a_i,n}^2 \sigma_G^2 + m_{a_i,n}^2)(\Gamma_3^{a_i,n})^2 \\
&\quad + 2(\mu_n \overline{\nu}_{a_i,n} \sigma_G^2 + w(a_i) m_{a_i,n}^2)(\Gamma_2^{a_i,n} - \Gamma_2^n)\Gamma_3^{a_i,n}.
\end{aligned}$$

By definition we have $m_{a_i,n} = (\Delta a_i)^{1/2}$ and $\overline{\nu}_{a_i,n} = \nu(a_i)\Delta a_i$. Moreover, we know from Lemma C.6 that $\Gamma_1^{a_i,n} - \Gamma_1^n$ and $\Gamma_2^{a_i,n} - \Gamma_2^n$ both are of order Δa_i. Hence, $\eta_{\Lambda\Lambda}^{a_i,n}$ is of the form $\alpha \Delta a_i + \beta (\Delta a_i)^2$ for some constants $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. On the other hand, we know that (see again (E.1))

$$\begin{aligned}
\eta_{V\Lambda}^{a_i,n} &= \begin{pmatrix} s_D + s_G \Gamma_1^n & s_G \Gamma_2^n & 0 \end{pmatrix} \boldsymbol{Q}^{a_i,n} \boldsymbol{Q}^{a_i,n} \left(\mathbf{\Gamma}^{a_i,n} - \begin{pmatrix}\mathbf{\Gamma}^n & 0\end{pmatrix}\right)^\mathsf{T} \\
&= \sigma_D^2 (s_D + s_G \Gamma_1^n)(\Gamma_1^{a_i,n} - \Gamma_1^n) + (\mu_n^2 \sigma_G^2 + \sigma_{\xi,n}^2) s_G \Gamma_2^n (\Gamma_2^{a_i,n} - \Gamma_2^n) \\
&\quad + (\mu_n \overline{\nu}_{a_i,n} \sigma_G^2 + w(a_i) m_{a_i,n}^2) s_G \Gamma_2^n \Gamma_3^{a_i,n},
\end{aligned}$$

which implies that $\eta_{V\Lambda}^{a_i,n}$ is of the form $\gamma \Delta a_i$ for a constant $\gamma \in \mathbb{R}$. This implies that $\lim_{n\to\infty} \frac{\eta_{V\Lambda}^{a_i,n}}{\sqrt{\eta_{\Lambda\Lambda}^{a_i,n}}} = 0$. ■

Proposition E.20. *Assume that Assumptions (A.E.1)–(A.E.4) hold true, and that Assumptions (A.E.10)–(A.E.12) of Theorem E.7 are satisfied as well. Consider the rescaled problem where the variable y is replaced by $z(\eta_{\Lambda\Lambda}^{a_i,n})^{1/2}$, and let $\tilde{J}_n(t,x,z) := J_n(t,x,z(\eta_{\Lambda\Lambda}^{a_i,n})^{1/2})$. Then a limiting solution $\tilde{J}$ in the sense of Theorem E.7 exists, and satisfies*

$$\tilde{J}(t,x,z) = \mathrm{e}^{-rt} \frac{1}{r} U(a_i, rx).$$

The feedback control of the limiting problem are $\tilde{c}^*(t,x,z) = rx$ and $\tilde{h}^*(t,x,z) = 0$.

Proof. Instead of solving the HJB equation (E.2) directly in (t,x,y), let us do a change of variables. In particular, we set $z = \delta^{a_i,n} y$ with $\delta^{a_i,n} = (\eta^{a_i,n}_{\Lambda\Lambda})^{-1/2}$, so that (E.2) becomes

$$
\begin{aligned}
0 = &-\tilde J_{n,t}(t,x,z) - rx\,\tilde J_{n,x}(t,x,z) - b_n\big(\tfrac{z}{\delta^{a_i,n}}\big)\delta^{a_i,n}\,\tilde J_{n,z}(t,x,z) - \frac{\eta^{a_i,n}_{\Lambda\Lambda}}{2}(\delta^{a_i,n})^2\tilde J_{n,zz}(t,x,z)\\
&- \sup_{c\in\mathbb{R}}\Big(e^{-rt}U(a_i,c) - c\,\tilde J_{n,x}(t,x,z)\Big)\\
&- \sup_{h\in\mathbb{R}}\Big(h\,m_n\big(t,\tfrac{z}{\delta^{a_i,n}}\big)\,\tilde J_{n,x}(t,x,z) + h\,\eta^{a_i,n}_{V\Lambda}\delta^{a_i,n}\tilde J_{n,xz}(t,x,z) + \frac{1}{2}h^2\eta^{a_i,n}_{VV}\,\tilde J_{n,xx}(t,x,z)\Big),
\end{aligned}
$$

where $\tilde J_n(t,x,z) = J_n(t,x,z/\delta^{a_i,n})$. Since by definition $b_n(y) = (\bar\beta^n(a_i) - \alpha^n_\Lambda)y$ we find that $b_n(\frac{z}{\delta^{a_i,n}})\delta^{a_i,n} = b_n(z)$. However, in contrast to the proof of Lemma E.18, it now holds that (see Lemma E.19)

$$
\lim_{n\to\infty}\eta^{a_i,n}_{\Lambda\Lambda}(\delta^{a_i,n})^2 = 1, \qquad \lim_{n\to\infty} m_n\big(t,\tfrac{z}{\delta^{a_i,n}}\big) = 0, \qquad \lim_{n\to\infty}\eta^{a_i,n}_{V\Lambda}\delta^{a_i,n} = 0.
$$

Hence, if the Assumptions (A.E.10)–(A.E.12) of Theorem E.7 are satisfied there exists a unique continuous viscosity solution $\tilde J$ satisfying the limiting HJB equation

$$
F(t,x,z,\tilde J_t, D\tilde J, D^2\tilde J) = 0,
$$

with the Bellman operator F satisfying

$$
F(t,x,z,q,p,X) := -q - rx\,p_x + \alpha_\Lambda z\,p_z - \tfrac{1}{2}X_{zz} - \sup_{c\in\mathbb{R}}\big(\tfrac{U(a_i,c)}{\mathrm{e}^{rt}} - c\,p_x\big) - \sup_{h\in\mathbb{R}}\tfrac{h^2}{2}\eta_{VV}\,X_{xx}.
$$

More explicitly, we have

$$
\tilde J_t = -rx\,\tilde J_x + \alpha_\Lambda z\,\tilde J_z - \frac{1}{2}\tilde J_{zz} - \sup_{c\in\mathbb{R}}\Big(\mathrm{e}^{-rt}\,U(a_i,c) - c\,\tilde J_x\Big) - \sup_{h\in\mathbb{R}}\frac{h^2}{2}\eta_{VV}\,\tilde J_{xx},
$$

and $\tilde J_n \to \tilde J$ locally uniformly on $[0,\infty)\times E$.

It therefore suffices to exhibit *one* solution of $F(t,x,z,\tilde J_t, D\tilde J, D^2\tilde J) = 0$ inside the uniqueness class of Theorem E.7. For this purpose, let

$$
\widehat J(t,x,z) := \mathrm{e}^{-rt}\,\frac{1}{r}U(a_i,rx),
$$

which is of class $C^{1,2}$ with $\widehat J_z = \widehat J_{zz} = \widehat J_{xz} = 0$, and

$$
\widehat J_t = -\,\mathrm{e}^{-rt}\,U(a_i,rx), \qquad \widehat J_x = \mathrm{e}^{-rt}\,U'(a_i,rx), \qquad \widehat J_{xx} = r\,\mathrm{e}^{-rt}\,U''(a_i,rx) < 0.
$$

By strict concavity of $U(a_i,\cdot)$ and (A.E.4), the supremum over c in $F(t,x,z,\tilde J_t, D\tilde J, D^2\tilde J) = 0$

is attained at the unique c with $\mathrm{e}^{-rt}\, U'(a_i, c) = \widehat{J}_x$, i.e. at $c = rx$, with value

$$\mathrm{e}^{-rt}\, U(a_i, rx) - rx\, \mathrm{e}^{-rt}\, U'(a_i, rx).$$

Moreover, since $\widehat{J}_{xx} < 0$, the supremum over h is attained at $h = 0$ with value 0. Substituting, these expressions back into $F(t, x, z, \tilde{J}_t, D\tilde{J}, D^2\tilde{J}) = 0$ finally yields

$$\begin{aligned} &F\big(t, x, z, \widehat{J}_t, D\widehat{J}, D^2\widehat{J}\big) \\ &\qquad = \mathrm{e}^{-rt}\, U(a_i, rx) - rx\ \mathrm{e}^{-rt}\, U'(a_i, rx) - \Big(\mathrm{e}^{-rt}\, U(a_i, rx) - rx\ \mathrm{e}^{-rt}\, U'(a_i, rx)\Big) = 0. \end{aligned}$$

Hence $\widehat{J}$ is a classical, and thus a viscosity solution of $F(t, x, z, \tilde{J}_t, D\tilde{J}, D^2\tilde{J}) = 0$, with maximizers $\tilde{c}^*(t, x, z) = rx$ and $\tilde{h}^*(t, x, z) = 0$.

It remains to check that $\widehat{J}$ lies in the class in which Theorem E.7 asserts uniqueness, i.e., that it satisfies conditions (i)–(iii) of Lemma E.14. Since $\mathrm{e}^{rt}\, |\widehat{J}(t, x, z)| = \frac{1}{r}|U(a_i, rx)|$ does not depend on z, it is bounded on $[0, \infty) \times I_x \times \mathbb{R}$ for every compact $I_x \subset (0, \infty)$ and $\varrho \geq 1$, which gives (i) and (iii). For (ii), fix $T > 0$ and a compact $K \subset (0, \infty) \times \mathbb{R}$ and let I_x be its projection onto the x-axis. Then $c_{T,K} := \min_{x \in I_x} \mathrm{e}^{-rT}\, U'(a_i, rx) > 0$ and $\lambda_{T,K} := \min_{x \in I_x} \big(- r\, \mathrm{e}^{-rT}\, U''(a_i, rx)\big) > 0$, so $x \mapsto \widehat{J} - c_{T,K} x$ is non-decreasing and $x \mapsto \widehat{J} + \frac{\lambda_{T,K}}{2} x^2$ is concave on the x-sections of K, for every $t \in [0, T]$. Hence Lemma E.14 applies to the pair $(\tilde{J}, \widehat{J})$ in both orders and yields $\tilde{J} = \widehat{J}$. ■

In order to connect the limiting solution $\tilde{J}(t, x, z)$ from Proposition E.20 to the value function $J(t, x, y)$ of the limiting HJB equation of Theorem E.7, we need the following result.

Lemma E.21. *Let Assumptions (A.E.1)–(A.E.4) and (A.E.10)–(A.E.12) of Theorem E.7 hold. Then a limiting solution $J(t, x, y)$ to the original HJB equation* (E.2) *in the sense of Theorem E.7 exists and satisfies the following assertions:*

(1) $J(t, x, y) = J(t, x, -y)$ for all $(t, x) \in [0, \infty) \times (0, \infty)$;

(2) $J(t, x, 0) = \mathrm{e}^{-rt}\, \frac{1}{r} U(a_i, rx)$ for all $(t, x) \in [0, \infty) \times (0, \infty)$.

Moreover, if in addition Assumption (A.E.13) holds, so that $J \in C^{1,2}([0, \infty) \times E)$ by Proposition E.16, then

(3) for all $(t, x) \in [0, \infty) \times (0, \infty)$, it holds that

$$J_x(t, x, 0) = \frac{U'(a_i, rx)}{\mathrm{e}^{rt}}, \quad J_{xx}(t, x, 0) = \frac{rU''(a_i, rx)}{\mathrm{e}^{rt}} < 0, \quad J_y(t, x, 0) = J_{xy}(t, x, 0) = 0,$$

and consequently

$$\frac{J_x(t, x, 0)}{J_{xx}(t, x, 0)} = -\frac{1}{r\psi(a_i, rx)}, \qquad \frac{J_{xy}(t, x, 0)}{J_{xx}(t, x, 0)} = 0.$$

Proof. Together with Lemma E.18 we may apply Theorem E.7, so that the limiting HJB equation reads as

$$\begin{aligned}0 &= -J_t(t,x,y) - rx\,J_x(t,x,y) - \alpha_\Lambda y\,J_y(t,x,y)\\ &\quad - \sup_{c\in\mathbb{R}}\Big(e^{-rt}U(a_i,c) - c\,J_x(t,x,y)\Big)\\ &\quad - \sup_{h\in\mathbb{R}}\Big(h\,\beta(a_i)y\,J_x(t,x,y) + \frac{1}{2}h^2\eta_{VV}J_{xx}(t,x,y)\Big).\end{aligned}$$

Now, the substitution $y' = -y$ yields $\frac{\partial}{\partial y} = -\frac{\partial}{\partial y'}$, and therefore

$$-y'\frac{\partial}{\partial y'}J(x,y') = y\frac{\partial}{\partial y'}J(x,y') = -y\frac{\partial}{\partial y}J(x,-y),$$

which together with

$$\begin{aligned}&\sup_{h\in\mathbb{R}}\Big(h\,s_G r y'\,J_x(t,x,y') + \frac{1}{2}h^2\eta_{VV}J_{xx}(t,x,y')\Big)\\ &\quad = \sup_{h\in\mathbb{R}}\Big((-h)\,s_G r y\,J_x(t,x,-y) + \frac{1}{2}(-h)^2\eta_{VV}J_{xx}(t,x,-y)\Big),\end{aligned}$$

implies that the HJB equation is invariant under the substitution $y' = -y$, and thus we have $J(t,x,y) = J(t,x,-y)$. Since the conditions (i)–(iii) of Lemma E.14 are conditions on the x-sections, uniform over compacts in y, the function J^-, defined via $J^-(t,x,y) = J(t,x,-y)$, lies in the same uniqueness class as J. Uniqueness in Theorem E.7 therefore gives $J^- = J$, which is assertion (1).

By construction $\tilde{J}_n(t,x,z) = J_n(t,x,z(\eta_{\Lambda\Lambda}^{a_i,n})^{1/2})$, and in particular

$$\tilde{J}_n(t,x,0) = J_n(t,x,0) \qquad \text{for every } n \in \mathbb{N}.$$

Theorem E.7 gives $J_n \to J$ locally uniformly, so $J_n(t,x,0) \to J(t,x,0)$. Proposition E.20 gives $\tilde{J}_n \to \tilde{J}$ locally uniformly, so $\tilde{J}_n(t,x,0) \to \tilde{J}(t,x,0) = \mathrm{e}^{-rt}\,\frac{1}{r}U(a_i,rx)$. The two limits of one and the same sequence coincide, which is assertion (2).

Under Assumption (A.E.13), Proposition E.16 gives $J \in C^{1,2}([0,\infty)\times E)$. Assertion (2) is an identity between two functions of x on the open set $(0,\infty)$, so it may be differentiated in x, which yields $J_x(t,x,0) = \mathrm{e}^{-rt}\,U'(a_i,rx)$ and $J_{xx}(t,x,0) = r\,\mathrm{e}^{-rt}\,U''(a_i,rx)$; the latter is strictly negative by strict concavity of $U(a_i,\cdot)$. By assertion (1) the map $y \mapsto J(t,x,y)$ is even and differentiable, hence $J_y(t,x,0) = 0$ for every $x \in (0,\infty)$; differentiating this identity in x gives $J_{xy}(t,x,0) = 0$. Finally, we obtain

$$\frac{J_x(t,x,0)}{J_{xx}(t,x,0)} = \frac{\mathrm{e}^{-rt}\,U'(a_i,rx)}{r\,\mathrm{e}^{-rt}\,U''(a_i,rx)} = \frac{U'(a_i,rx)}{rU''(a_i,rx)} = -\frac{1}{r\psi(a_i,rx)},$$

which concludes the proof. ■

Corollary E.22. *Let the assumptions from Proposition E.16 hold, and let J be as in Lemma E.21. In addition, assume that for each fixed $t \geq 0$ it holds $(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \to (v_0(a_i), 0)$ in probability as $n \to \infty$. Then, it holds that*

$$\frac{J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})} \to \frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)} = \frac{U'(a_i, rv_0(a_i))}{rU''(a_i, rv_0(a_i))} = -\frac{1}{r\psi(a_i, rv_0(a_i))},$$

as well as

$$\frac{J_{xy}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})} \to \frac{J_{xy}(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)} = 0,$$

both in probability as $n \to \infty$. If in addition each of the two sequences of ratios

$$\left(\frac{J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}\right)_{n\in\mathbb{N}} \qquad \text{and} \qquad \left(\frac{J_{xy}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}\right)_{n\in\mathbb{N}}$$

is uniformly integrable, then the convergence holds true in $L^1(P)$.

Proof. For each $M > 0$, we define the compact subset $K_M \subseteq E = (0, \infty) \times \mathbb{R}$ via

$$K_M := \left\{(x, y) \in E \colon \tfrac{1}{M} \leq x \leq M,\ |y| \leq M\right\}.$$

For simplicity we further denote

$$X_t^{a_i,n} := \frac{J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}, \qquad X_t^{a_i} := \frac{J_x(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}, \qquad x_t^{a_i} := \frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)}.$$

For a fixed $\eta > 0$ and $M_0 > 0$ with $(v_0(a_i), 0) \in \operatorname{int}(K_{M_0})$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds $P\big[(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \notin K_{M_0}\big] \leq \eta$. This fact is a direct consequence of $(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \to (v_0(a_i), 0)$ in probability as $n \to \infty$ for each fixed $t \geq 0$. On the other hand, for each of the finitely many $n < N$ the random variable $(V_t^{a_i,n}, \Lambda_t^{a_i,n})$ takes values in E almost surely, and we have $K_M \nearrow E$. Hence there is $M \geq M_0$ with $P\big[(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \notin K_M\big] \leq \eta$ for all $n < N$, and because $K_{M_0} \subseteq K_M$ it holds $\sup_{n\in\mathbb{N}} P\big[(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \notin K_M\big] \leq \eta$.

Now, on the event $\{(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \in K_M\}$ we have

$$|X_t^{a_i,n} - X_t^{a_i}| \leq \sup_{(t,x,y)\in[0,T]\times K_M} \left|\frac{J_x^{a_i,n}(t,x,y)}{J_{xx}^{a_i,n}(t,x,y)} - \frac{J_x(t,x,y)}{J_{xx}(t,x,y)}\right| =: \delta_{T,M},$$

and therefore, for an arbitrary $\varepsilon > 0$, it holds that

$$P\big[|X_t^{a_i,n} - X_t^{a_i}| > \varepsilon\big] \leq P[\delta_{T,M} > \varepsilon] + P\big[(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \notin K_M\big].$$

Since K_M is a compact subset of E, it follows from Corollary E.17 that $\delta_{T,M} \to 0$ as $n \to \infty$

for each fixed M. Hence, given $\eta > 0$, choose $M > 0$ such that $P\big[(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \notin K_M\big] \leq \eta$, so that $\limsup_{n\to\infty} P\big[|X_t^{a_i,n} - X_t^{a_i}| > \varepsilon\big] \leq \eta$. Since $\eta > 0$ is arbitrary let $\eta \searrow 0$, which eventually yields $\lim_{n\to\infty} P\big[|X_t^{a_i,n} - X_t^{a_i}| > \varepsilon\big] = 0$.

On the other hand, we have by Proposition E.16 that $J \in C^{1,2}$ with $J_{xx} \leq -\lambda_{T,K} < 0$ on compact subsets $[0,T] \times K$, which implies that $\frac{J_x}{J_{xx}}$ is continuous on $[0,\infty) \times E$. Since $(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \to (v_0(a_i), 0)$ in probability as $n \to \infty$ for each fixed $t \geq 0$ by assumption, it therefore follows by continuity $\frac{J_x}{J_{xx}}$ that

$$\lim_{n\to\infty} P\big[|X_t^{a_i} - x_t^{a_i}| > \varepsilon\big] = \lim_{n\to\infty} P\left[\left|\frac{J_x(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})} - \frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)}\right| > \varepsilon\right] = 0.$$

Since $|X_t^{a_i,n} - x_t^{a_i}| \leq |X_t^{a_i,n} - X_t^{a_i}| + |X_t^{a_i} - x_t^{a_i}|$ this is enough to prove that

$$\left|\frac{J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})} - \frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)}\right| \to 0$$

in probability as $n \to \infty$. The argument for $\frac{J_{xy}}{J_{xx}}$ is analogue.

To conclude the proof, let us first recall from Lemma E.21 that

$$J_x(t,x,0) = \mathrm{e}^{-rt}\, U'(a_i, rx), \qquad J_{xx}(t,x,0) = r\,\mathrm{e}^{-rt}\, U''(a_i, rx), \qquad J_{xy}(t,x,0) = 0$$

for every $x \in (0,\infty)$. Evaluating at $v_0(a_i)$ yields

$$\frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)} = \frac{U'(a_i, rv_0(a_i))}{rU''(a_i, rv_0(a_i))} = -\frac{1}{r\psi(a_i, rv_0(a_i))}, \qquad \frac{J_{xy}(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)} = 0.$$

To conclude the proof, note that convergence in probability together with uniform integrability of the family of ratios gives convergence in $L^1(P)$. ■

E.9. Summary

To conclude this section, let us give a brief overview of the assumptions we have used throughout Appendix E. The table below summarizes the assumption numbers and shows their main purpose including the first result they are used.

Assumption	Main purpose	First occurrence	Dependency
(A.E.1)	Existence of viscosity solutions	Proposition E.1	Utility function
(A.E.2)			Model
(A.E.3)			Value function
(A.E.4)	Feedback formulas	Proposition E.1	Utility function
(A.E.5)	Verification of optimality	Proposition E.3	Model & $U(a_i, \cdot)$
(A.E.6)			Model & $U(a_i, \cdot)$
(A.E.7)			Model & $U(a_i, \cdot)$
(A.E.8)	Convergence of viscosity solutions	Theorem E.7	Model
(A.E.9)			Model
(A.E.10)			Value function
(A.E.11)			Value function
(A.E.12)			Value function
(A.E.13)	Convergence of partials	Proposition E.16	Value function

As can be seen from the above table, there are essentially four groups of assumptions. The first group is (A.E.1)–(A.E.3) which is used to derive the existence of a viscosity solution to the HJB equation in the discrete case. Assumption (A.E.4) is used to invoke the feedback formulas. The second group (A.E.5)–(A.E.7) is used to verify that the existing candidate is indeed optimal. The third group is (A.E.8)–(A.E.12), which ensures that the limiting HJB equation has a solution and that the discrete solutions converge to it. Finally, (A.E.13) is used for classical solutions and ensures that the solution as well as its derivatives converge to a classical solution in the limit.

We prove in Lemma E.18 that (A.E.8) and (A.E.9) are satisfied in our setup. Moreover, (A.E.2) is also satisfied in our setup. The remaining assumptions, namely (A.E.1), (A.E.3)–(A.E.7), and (A.E.10)–(A.E.13) depend either explicitly or implicitly on the choice of the utility function U, and can thus only be checked on a case by case basis. In fact, (A.E.1) and (A.E.4) explicitly depend on U, and can thus be checked easily. However, all of the remaining assumptions depend on the solution to the HJB equation (E.2) in one way or the other and can therefore not be checked directly.

F. Market clearing

In this section we give the proofs of all the results in Section 3. Section F.1 covers the proof about the aggregate demand function (3.3), and we give the proof of the convergence of $n\,\mathrm{E}[(H_t^{a_i,n})^*]$ and $n\,\mathrm{Var}[(H_t^{a_i,n})^*]$ under an additional assumption. In Section F.2 we discuss the risk premium. In Section F.3 we give the proofs of Proposition 3.1 and Proposition 3.2 which characterize the weight functions in a trivial and non-trivial equilibrium, respectively.

Finally, in Section F.4 we give the existence proof that a non-trivial equilibrium exists.

F.1. Demand function

The following result is used in the main text to provide some interpretation about the optimal trading strategy in the continuous economy.

Proposition F.1. *Assume that $J^{a_i,n}$ is a classical solution to the HJB equation* (E.2)*. Further, assume that Assumptions (A.E.1)–(A.E.4) hold true, and that Assumptions (A.E.10)–(A.E.12) of Theorem E.7 are satisfied as well. Then, the optimal trading strategy of investor a_i in the n-th discrete economy is of the form*

$$(H_t^{a_i,n})^* = -\frac{(-\frac{r\rho}{n} + \beta^n(a_i)\Lambda_t^{a_i,n})J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n}) + \eta_{V\Lambda}^{a_i,n} J_{xy}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{\eta_{VV}^{a_i,n} J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}.$$

Under the additional Assumption (A.E.13), and if the following three conditions hold true for each $t \geq 0$:

- *(i)* $(V_t^{a_i,n}, \Lambda_t^{a_i,n}) \to (v_0(a_i), 0)$ *in probability as* $n \to \infty$*;*
- *(ii)* $\sup_{n\in\mathbb{N}} \left(\left\| \frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})} \right\|_{L^4(P)} + \left\| \frac{J_{xy}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})} \right\|_{L^4(P)} \right) < \infty$*;*
- *(iii)* $\lim_{n\to\infty} \sqrt{n} \left\| \frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})} - \frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)} \right\|_{L^2(P)} = 0$*;*

the optimal trading strategy $(H_t^{a_i,n})^*$ has the following limiting properties:

- *(1)* $\lim_{n\to\infty} n\,\mathrm{E}[(H_t^{a_i,n})^*] = \frac{\theta \frac{1}{\psi(a_i,rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(a,rv_0(a))}\,\mathrm{d}a}$ *and* $\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \frac{w(a_i)^2}{\sigma_\xi^2}\mathcal{V}_{(1,t)}^2 \frac{1-\mathrm{e}^{-2\alpha_\Lambda t}}{2\alpha_\Lambda}$.
- *(2) The random variable* $\frac{1}{\sqrt{n}}\Big(n(H_t^{a_i,n})^* - \frac{\theta \frac{1}{\psi(a_i,rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(a,rv_0(a))}\,\mathrm{d}a}\Big)$ *converges in distribution to a normal distribution with mean zero and variance* $\frac{w(a_i)^2}{\sigma_\xi^2}\mathcal{V}_{(1,t)}^2 \frac{1-\mathrm{e}^{-2\alpha_\Lambda t}}{2\alpha_\Lambda}$.
- *(3) In steady state,*

$$\frac{1}{\sqrt{n}} \left(n(H_t^{a_i,n})^* - \frac{\theta \frac{1}{\psi(a_i,rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(u,rv_0(u))}\,\mathrm{d}u} \right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, w^2(a_i)\frac{\mathcal{V}_{(1,t)}^2}{2\alpha_\Lambda \sigma_\xi^2}\right).$$

Proof. As in the proof of Corollary E.22 we denote

$$X_t^{a_i,n} := \frac{J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}, \qquad X_t^{a_i} := \frac{J_x(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}, \qquad x_t^{a_i} := \frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)},$$

and here additionally

$$Y_t^{a_i,n} := \frac{J_{xy}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}, \qquad Y_t^{a_i} := \frac{J_{xy}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}.$$

By assumption (i) it follows from Corollary E.22 that $X_t^{a_i,n} \to x_t^{a_i}$ and $Y_t^{a_i,n} \to 0$ in probability as $n \to \infty$ for each fixed $t \geq 0$. Together with assumption (ii) the families of ratios $(X_t^{a_i,n})_{n\in\mathbb{N}}$ and $(Y_t^{a_i,n})_{n\in\mathbb{N}}$ are uniformly integrable (see Theorem 13.3 in Williams (1991)). Moreover, Hölder's inequality with exponents $\frac{4}{p}$ and $1/(1-\frac{p}{4}) = \frac{4}{4-p}$ implies

$$\begin{aligned}
\mathrm{E}[|X_t^{a_i,n} - x_t^{a_i}|^p] &\leq \varepsilon^p + \mathrm{E}\left[|X_t^{a_i,n} - x_t^{a_i}|^p \mathbb{1}_{\{|X_t^{a_i,n} - x_t^{a_i}| > \varepsilon\}}\right] \\
&\leq \varepsilon^p + \mathrm{E}\left[|X_t^{a_i,n} - x_t^{a_i}|^4\right]^{p/4} P\left[|X_t^{a_i,n} - x_t^{a_i}| > \varepsilon\right]^{1-p/4} \\
&\leq \varepsilon^p + \|X_t^{a_i,n} - x_t^{a_i}\|_{L^4(P)}^p P\left[|X_t^{a_i,n} - x_t^{a_i}| > \varepsilon\right]^{1-p/4},
\end{aligned}$$

so that assumptions (i) and (ii) further imply that $X_t^{a_i,n} \to x_t^{a_i}$ in $L^p(P)$ for every $p < 4$. The same holds true for $Y_t^{a_i,n} \to 0$ by the exact same argumentation.

Proof of property (1): We first note that

$$n\,\mathrm{E}[(H_t^{a_i,n})^*] = \frac{r\rho}{\eta_{VV}^{a_i,n}}\,\mathrm{E}[X_t^{a_i,n}] + \frac{\beta^n(a_i)}{\eta_{VV}^{a_i,n}} n\,\mathrm{E}[\Lambda_t^{a_i,n} X_t^{a_i,n}] + \frac{\eta_{V\Lambda}^{a_i,n}}{\eta_{VV}^{a_i,n}} n\,\mathrm{E}[Y_t^{a_i,n}].$$

By the argumentation above we know that $\mathrm{E}[X_t^{a_i,n}] \to x_t^{a_i}$, so that

$$\frac{r\rho}{\eta_{VV}^{a_i,n}}\,\mathrm{E}[X_t^{a_i,n}] \to \frac{r\rho}{\eta_{VV}} \cdot \frac{J_x(t, v_0(a_i), 0)}{J_{xx}(t, v_0(a_i), 0)} = \frac{\rho}{\eta_{VV}\psi(a_i, rv_0(a_i))},$$

since $\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)} = -\frac{1}{r\psi(a_i,rv_0(a_i))}$ by Corollary E.22, and where we denote $\eta_{VV} = \lim_{n\to\infty} \eta_{VV}^{a_i,n}$. For the second term, we use that $\mathrm{E}[\Lambda_t^{a_i,n}] = 0$, so that

$$\mathrm{E}[\Lambda_t^{a_i,n} X_t^{a_i,n}] = \mathrm{E}[\Lambda_t^{a_i,n}(X_t^{a_i,n} - x_t^{a_i})] + \mathrm{E}[\Lambda_t^{a_i,n}]x_t^{a_i} = \mathrm{E}[\Lambda_t^{a_i,n}(X_t^{a_i,n} - x_t^{a_i})],$$

and therefore

$$\left|n\,\mathrm{E}[\Lambda_t^{a_i,n} X_t^{a_i,n}]\right| \leq \left(\sqrt{n}\,\mathrm{E}[|\Lambda_t^{a_i,n}|^2]^{1/2}\right)\left(\sqrt{n}\,\mathrm{E}[|X_t^{a_i,n} - x_t^{a_i}|^2]^{1/2}\right) \to 0$$

by assumption (iii) and the fact that $\mathrm{E}[(\Lambda_t^{a_i,n})^2] = \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}(1 - \mathrm{e}^{-2\alpha_\Lambda^n t})$ with $n\eta_{\Lambda\Lambda}^{a_i,n}$ converging to a constant as $n \to \infty$ (see below). Finally, the fact that $n\eta_{V\Lambda}^{a_i,n}$ converges to a constant as $n \to \infty$ (see the proof of Lemma E.19 together with Lemma C.6 and the definitions of $m_{a_i,n}$ and $\overline{\nu}_{a_i,n}$ in Appendix B), together with $\mathrm{E}[Y_t^{a_i,n}] \to 0$ implies $n\eta_{V\Lambda}^{a_i,n}\,\mathrm{E}[Y_t^{a_i,n}] \to 0$ as $n \to \infty$.

As we show in Proposition F.4, the risk premium ρ satisfies

$$\rho = -\frac{\eta_{VV}\theta}{\int_0^1 \frac{1}{\psi(a, rv_0(a))}\,\mathrm{d}a},$$

and therefore

$$\lim_{n\to\infty} n\,\mathrm{E}[(H_t^{a_i,n})^*] = \frac{\theta \frac{1}{\psi(a_i, rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(a, rv_0(a))}\,\mathrm{d}a}.$$

This proves the first part of property (1).

To analyze the second limit $\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*]$, we observe from previous calculations that $\lim_{n\to\infty} n\,\mathrm{E}[(H_t^{a_i,n})^*]^2 = 0$, and therefore $\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \lim_{n\to\infty} n\,\mathrm{E}[((H_t^{a_i,n})^*)^2]$. Consequently, it holds that

$$n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \frac{1}{n(\eta_{VV}^{a_i,n})^2}\,\mathrm{E}\left[\left((-r\rho + \beta^n(a_i) n\Lambda_t^{a_i,n})X_t^{a_i,n} + \eta_{V\Lambda}^{a_i,n} n Y_t^{a_i,n}\right)^2\right].$$

In fact, by expanding the square only one of the six different terms remains. Indeed, Minkowski's inequality yields

$$\mathrm{E}[(X_t^{a_i,n})^2] = \mathrm{E}[(X_t^{a_i,n} - x_t^{a_i} + x_t^{a_i})^2] \le \left(\mathrm{E}[(X_t^{a_i,n} - x_t^{a_i})^2]^{1/2} + \mathrm{E}[(x_t^{a_i})^2]^{1/2}\right)^2,$$

so that $\lim_{n\to\infty} \frac{r^2\rho^2}{n(\eta_{VV}^{a_i,n})^2}\,\mathrm{E}[(X_t^{a_i,n})^2] = 0$. Then, by the same argumentation as above we have

$$n(\eta_{V\Lambda}^{a_i,n})^2\,\mathrm{E}[(Y_t^{a_i,n})^2] = \sqrt{n}(\eta_{V\Lambda}^{a_i,n})^2\sqrt{n}\,\mathrm{E}[(Y_t^{a_i,n})^2] \to 0,$$

which implies $\lim_{n\to\infty} \frac{n^2(\eta_{V\Lambda}^{a_i,n})^2\,\mathrm{E}[(Y_t^{a_i,n})^2]}{n(\eta_{VV}^{a_i,n})^2} = 0$. Next, it holds

$$|\eta_{V\Lambda}^{a_i,n}\,\mathrm{E}[X_t^{a_i,n}Y_t^{a_i,n}]| \le |\eta_{V\Lambda}^{a_i,n}|\,\mathrm{E}[(X_t^{a_i,n})^2]^{1/2}\,\mathrm{E}[(Y_t^{a_i,n})^2]^{1/2} \to 0 \qquad \text{as } n\to\infty,$$

so that $\lim_{n\to\infty} \frac{-2r\rho}{(\eta_{VV}^{a_i})^2}\,\mathrm{E}[X_t^{a_i,n}Y_t^{a_i,n}] = 0$. Similarly, we have

$$|n\eta_{V\Lambda}^{a_i,n}\,\mathrm{E}[\Lambda_t^{a_i,n}X_t^{a_i,n}Y_t^{a_i,n}]| \le n|\eta_{V\Lambda}^{a_i,n}|\,\mathrm{E}[(\Lambda_t^{a_i,n})^2]^{1/2}\,\mathrm{E}[(X_t^{a_i,n})^4]^{1/4}\,\mathrm{E}[(Y_t^{a_i,n})^4]^{1/4},$$

which together with assumption (ii) and the fact that $n\eta_{V\Lambda}^{a_i,n}$ converges to a constant and $\mathrm{E}[(\Lambda_t^{a_i,n})^2]^{1/2} \to 0$ implies $\lim_{n\to\infty} \frac{2n^2\beta^n(a_i)\eta_{V\Lambda}^{a_i,n}\,\mathrm{E}[\Lambda_t^{a_i,n}X_t^{a_i,n}Y_t^{a_i,n}]}{n(\eta_{VV}^{a_i,n})^2} = 0$. Finally, from assumption (ii), the fact that $\mathrm{E}[(\Lambda_t^{a_i,n})^2]^{1/2} \to 0$, and because

$$|\,\mathrm{E}[\Lambda_t^{a_i,n}(X_t^{a_i,n})^2]| \le \mathrm{E}[(\Lambda_t^{a_i,n})^2]^{1/2}\,\mathrm{E}[(X_t^{a_i,n})^4]^{1/2},$$

it follows that $\lim_{n\to\infty} \frac{-2r\rho n\beta^n(a_i)\,\mathrm{E}[\Lambda_t^{a_i,n}(X_t^{a_i,n})^2]}{n(\eta_{VV}^{a_i,n})^2} = 0$. Consequently, it holds that

$$\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \lim_{n\to\infty} \frac{\beta^n(a_i)^2}{(\eta_{VV}^{a_i,n})^2} n\,\mathrm{E}\left[(\Lambda_t^{a_i,n})^2(X_t^{a_i,n})^2\right].$$

Since $\sqrt{n}\Lambda_t^{a_i,n} \sim \mathcal{N}(0, n\frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}(1-\mathrm{e}^{-2\alpha_\Lambda^n t}))$, it holds that

$$\mathrm{E}[(\sqrt{n}\Lambda_t^{a_i,n})^6] = 15\,\mathrm{E}[(\sqrt{n}\Lambda_t^{a_i,n})^2]^3 = 15\left(n\frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}(1-\mathrm{e}^{-2\alpha_\Lambda^n t})\right)^3,$$

and therefore

$$\left|n\,\mathrm{E}\left[(\Lambda_t^{a_i,n})^2\big((X_t^{a_i,n})^2 - (x_t^{a_i})^2\big)\right]\right| \le n\,\mathrm{E}[(\Lambda_t^{a_i,n})^6]^{1/3}\,\mathrm{E}[|X_t^{a_i,n} - x_t^{a_i}|^3]^{1/3}\,\mathrm{E}[|X_t^{a_i,n} + x_t^{a_i}|^3]^{1/3}$$

implies together with the discussion at the beginning of the proof that

$$\lim_{n\to\infty} n\,\mathrm{E}\left[(\Lambda_t^{a_i,n})^2\big((X_t^{a_i,n})^2 - (x_t^{a_i})^2\big)\right] = 0,$$

and thus

$$\lim_{n\to\infty} n\,\mathrm{E}\left[(\Lambda_t^{a_i,n})^2(X_t^{a_i,n})^2\right] = \lim_{n\to\infty} n\,\mathrm{E}\left[(\Lambda_t^{a_i,n})^2\right](x_t^{a_i})^2.$$

By the definition of $\eta_{\Lambda\Lambda}^{a_i,n}$ given in (E.1), it follows from Lemma C.6 and Lemma C.12 that

$$\lim_{n\to\infty} n\eta_{\Lambda\Lambda}^{a_i,n} = \frac{(\nu(a_i)\sigma_\xi^2 - w(a_i)\mu)^2(\sigma_G^2 - \alpha_G\gamma)^2}{(\mu^2\sigma_G^2 + \sigma_\xi^2)^2},$$

and therefore together with $\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)} = -\frac{1}{r\psi(a_i,rv_0(a_i))}$ by Corollary E.22 we find

$$\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \frac{\beta(a_i)^2}{\eta_{VV}^2 r^2\psi(a_i,rv_0(a_i))^2} \cdot \frac{(\nu(a_i)\sigma_\xi^2 - w(a_i)\mu)^2(\sigma_G^2 - \alpha_G\gamma)^2}{2\alpha_\Lambda(\mu^2\sigma_G^2 + \sigma_\xi^2)^2}(1-\mathrm{e}^{-2\alpha_\Lambda t}).$$

We thus observe that in a trivial equilibrium it holds that $\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = 0$ (see Proposition 3.1). In a non-trivial equilibrium, however, we know that (see Proposition 3.2)

$$(\nu(a_i)\sigma_\xi^2 - w(a_i)\mu) = \kappa w(a_i)\frac{\psi(a_i,rv_0(a_i))}{\beta(a_i)}, \qquad \kappa \neq 0,$$

and therefore

$$\lim_{n\to\infty} n\,\mathrm{Var}[(H_t^{a_i,n})^*] = \frac{w(a_i)^2}{2\alpha_\Lambda} \cdot \frac{\kappa^2(\sigma_G^2 - \alpha_G\gamma)^2}{\eta_{VV}^2 r^2(\mu^2\sigma_G^2 + \sigma_\xi^2)^2}(1-\mathrm{e}^{-2\alpha_\Lambda t}).$$

By using (4.1), we may conclude the proof of property (1).

Proof of property (2): Let us first note that

$$\sqrt{n}(H_t^{a_i,n})^* = -\frac{1}{(\eta_{VV}^{a_i,n})^2}\Bigg(\Big(-\frac{r\rho}{\sqrt{n}} + \beta^n(a_i)\sqrt{n}\Lambda_t^{a_i,n}\Big)X_t^{a_i,n} + \sqrt{n}\eta_{V\Lambda}^{a_i,n}Y_t^{a_i,n}\Bigg).$$

Together with Corollary E.22 it holds

$$\frac{1}{(\eta_{VV}^{a_i,n})^2}\Big(-\frac{r\rho}{\sqrt{n}}X_t^{a_i,n} + \sqrt{n}\eta_{V\Lambda}^{a_i,n}Y_t^{a_i,n}\Big) \to 0$$

in probability as $n \to \infty$. Moreover, again by Corollary E.22, we have

$$\frac{\beta^n(a_i)}{\eta_{VV}^{a_i,n}}X_t^{a_i,n} \to -\frac{\beta(a_i)}{\eta_{VV}} \cdot \frac{1}{r\psi(a_i, rv_0(a_i))}$$

in probability as $n \to \infty$. On the other hand, note that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{\theta\frac{1}{\psi(a_i,rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(a,rv_0(a))}\,\mathrm{d}a} = 0.$$

Since $\sqrt{n}\Lambda_t^{a_i,n} \sim \mathcal{N}(0, n\frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}(1-\mathrm{e}^{-2\alpha_\Lambda^n t}))$ converges in distribution to a centered Gaussian random variable with variance

$$\frac{(\nu(a_i)\sigma_\xi^2 - w(a_i)\mu)^2(\sigma_G^2 - \alpha_G\gamma)^2}{2\alpha_\Lambda(\mu^2\sigma_G^2 + \sigma_\xi^2)^2}(1-\mathrm{e}^{-2\alpha_\Lambda t}),$$

it follows by Slutsky's theorem that

$$\frac{1}{\sqrt{n}}\left(n(H_t^{a_i,n})^* - \frac{\theta\frac{1}{\psi(a_i,rv_0(a_i))}}{\int_0^1 \frac{1}{\psi(a,rv_0(a))}\,\mathrm{d}a}\right)$$

converges in distribution to a centered Gaussian random variable with variance

$$\frac{(\nu(a_i)\sigma_\xi^2 - w(a_i)\mu)^2(\sigma_G^2 - \alpha_G\gamma)^2}{2\alpha_\Lambda(\mu^2\sigma_G^2 + \sigma_\xi^2)^2}(1-\mathrm{e}^{-2\alpha_\Lambda t}) \cdot \frac{\beta(a_i)^2}{\eta_{VV}^2 r^2\psi(a_i, rv_0(a_i))^2}.$$

Again, the conclusion follows together with (4.1).

Proof of property (3): The result follows from (1) and (2) as $t \to \infty$. ■

Next we prove the market clearing equation (3.3).

Proposition F.2. *Let the assumptions of Proposition E.16 hold, and assume additionally*

that
$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left\|\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}-\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}\right\|_{L^2(P)}\to 0,$$
and
$$\sup_{0\le i\le n-1}\left|\frac{J_{xy}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}\right|\to 0$$
in probability as $n\to\infty$. Then the sum

$$X_{(a,t)}^n:=\sum_{\substack{0\le i\le n-1,\\ a_i<a}}(H_t^{a_i,n})^*,\qquad (a,t)\in[0,1)\times\mathbb{R}_+$$

converges to $X_{(a,t)}$ uniformly on compacts in probability as $n\to\infty$, where $X_{(a,t)}$ satisfies

$$\begin{aligned}X_{(a,t)}=&-\int_0^a\frac{\rho(1+\alpha_\Lambda t)}{\eta_{VV}\psi(u,rv_0(u))}\,\mathrm{d}u-\alpha_\Lambda\int_0^t X_{(a,s)}\,\mathrm{d}s\\&+\int_0^a\frac{\beta(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2}{\eta_{VV}r\psi(u,rv_0(u))}\,\mathrm{d}u\int_0^t\mathrm{d}f_1(D_s,\widehat{G}_s,\Xi_s)\\&+\int_{[(0,0),(a,t))}\frac{\beta(u)(\nu(u)\sigma_\xi^2-w(u)\mu)}{\eta_{VV}r\psi(u,rv_0(u))}\,\mathrm{d}f_2(\xi_{(u,s)},\widehat{G}_s,\Xi_s).\end{aligned}$$

The functions f_1 and f_2 are defined as in Theorem C.9.

Proof. For simplicity, let us denote

$$Y_t^{a_i,n}:=\frac{J_{xy}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})},\qquad Z_t^{a_i,n}:=\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})},\qquad z_t^{a_i}:=\frac{J_x(t,v_0(a_i),0)}{J_{xx}(t,v_0(a_i),0)}.$$

Since the optimal trading strategy in the n-th discrete economy is given by

$$(H_t^{a_i,n})^*=-\frac{(-\frac{r\rho}{n}+\beta^n(a_i)\Lambda_t^{a_i,n})J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})+\eta_{V\Lambda}^{a_i,n}J_{xy}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{\eta_{VV}^{a_i,n}J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})},$$

the reduced notational form reads as

$$(H_t^{a_i,n})^*=Z_t^{a_i,n}\frac{r\rho}{\eta_{VV}^{a_i,n}}\cdot\frac{1}{n}-Z_t^{a_i,n}\frac{\beta^n(a_i)\Lambda_t^{a_i,n}}{\eta_{VV}^{a_i,n}}-Y_t^{a_i,n}\frac{\eta_{V\Lambda}^{a_i,n}}{\eta_{VV}^{a_i,n}}.$$

Since $\eta_{VV}^{a_i,n}$ does not depend on a_i (see the proof of Lemma E.18), and by applying Hölder's inequality, we find that

$$\mathrm{E}\left[\Big|\sum_{\substack{0\le i\le n-1,\\ a_i<a}}(Z_t^{a_i,n}-z_t^{a_i})\frac{r\rho}{\eta_{VV}^{a_i,n}}\cdot\frac{1}{n}\Big|\right]\le\frac{r|\rho|}{\eta_{VV}^{a_i,n}}\cdot\frac{1}{n}\sum_{\substack{0\le i\le n-1,\\ a_i<a}}\|Z_t^{a_i,n}-z_t^{a_i}\|_{L^2(P)}.$$

By assumption we thus obtain for each fixed $t \geq 0$ that

$$\sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} (Z_t^{a_i,n} - z_t^{a_i}) \frac{r\rho}{\eta_{VV}^{a_i,n}} \cdot \frac{1}{n} \to 0$$

converges in $L^1(P)$ and thus in probability as $n \to \infty$. On the other hand, by continuity of $a \mapsto \frac{J_x(t,v_0(a),0)}{J_{xx}(t,v_0(a),0)}$ it follows for each fixed $t \geq 0$ that

$$\sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} z_t^{a_i} \frac{r\rho}{\eta_{VV}^{a_i,n}} \cdot \frac{1}{n} \to \frac{r\rho}{\eta_{VV}} \int_0^a \frac{J_x(t, v_0(a), 0)}{J_{xx}(t, v_0(a), 0)} \, \mathrm{d}a$$

in probability as $n \to \infty$, and thus by Corollary E.22 we eventually find that

$$\sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} \frac{J_x^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})} \cdot \frac{r\rho}{\eta_{VV}^{a_i,n}} \cdot \frac{1}{n} \to -\int_0^a \frac{\rho}{\eta_{VV}\psi(u, rv_0(u))} \, \mathrm{d}u$$

in probability as $n \to \infty$, for each fixed $t \geq 0$.

Similarly, we know from Corollary E.22 that $Y_t^{a_i,n} \to 0$ in probability, and by assumption now also $\sup_{0 \leq i \leq n-1} |Y_t^{a_i,n}| \to 0$ in probability as $n \to \infty$ for each fixed $t \geq 0$. Since

$$\Bigg| \sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} Y_t^{a_i,n} \frac{\eta_{V\Lambda}^{a_i,n}}{\eta_{VV}^{a_i,n}} \Bigg| \leq \sup_{\substack{0 \leq i \leq n-1, \\ a_i < a}} |Y_t^{a_i,n}| \sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} \frac{|\eta_{V\Lambda}^{a_i,n}|}{\eta_{VV}^{a_i,n}}$$

and because $\eta_{V\Lambda}^{a_i,n} = \tilde{\eta}_{V\Lambda}^{a_i,n} \Delta a_i$ for some constant $\tilde{\eta}_{V\Lambda}^{a_i,n} \to \tilde{\eta}_{V\Lambda}^{a_i}$ (see Lemma E.19) with $\tilde{\eta}_{V\Lambda}^{a_i}$ being the evaluation of some continuous function $\tilde{\eta}_{V\Lambda} \colon [0,1) \to \mathbb{R}$ at $a_i \in [0,1)$, it holds that

$$\sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} \frac{|\eta_{V\Lambda}^{a_i,n}|}{\eta_{VV}^{a_i,n}} = \sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} \frac{|\tilde{\eta}_{V\Lambda}^{a_i,n}|}{\eta_{VV}^{a_i,n}} \Delta a_i \to \int_0^a \frac{|\tilde{\eta}_{V\Lambda}(u)|}{\eta_{VV}} \, \mathrm{d}u < \infty$$

in probability. Therefore, we find that

$$\sum_{\substack{0 \leq i \leq n-1, \\ a_i < a}} \frac{J_{xy}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t, V_t^{a_i,n}, \Lambda_t^{a_i,n})} \cdot \frac{\eta_{V\Lambda}^{a_i,n}}{\eta_{VV}^{a_i,n}} \to 0$$

in probability as $n \to \infty$, for each fixed $t \geq 0$.

Finally, for the term including the inference wedge, let us first note that

$$\mathrm{E}\left[\left|\sum_{\substack{0\le i\le n-1,\\ a_i<a}}(Z_t^{a_i,n}-z_t^{a_i})\frac{\beta^n(a_i)\Lambda_t^{a_i,n}}{\eta_{VV}^{a_i,n}}\right|\right] \le \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \mathrm{E}\left[(Z_t^{a_i,n}-z_t^{a_i})^2\right]^{1/2}\frac{|\beta^n(a_i)|}{\eta_{VV}^{a_i,n}}\,\mathrm{E}\left[(\Lambda_t^{a_i,n})^2\right]^{1/2}$$
$$\le \sum_{\substack{0\le i\le n-1,\\ a_i<a}} \left\|Z_t^{a_i,n}-z_t^{a_i}\right\|_{L^2(P)}\frac{|\beta^n(a_i)|}{\eta_{VV}^{a_i,n}}\cdot\frac{(\eta_{\Lambda\Lambda}^{a_i,n})^{1/2}}{(2\alpha_\Lambda^n)^{1/2}}$$

where we use that

$$\mathrm{E}\left[(\Lambda_t^{a_i,n})^2\right] = \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}(1-\mathrm{e}^{-2\alpha_\Lambda^n t}) \le \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}.$$

Since $\eta_{\Lambda\Lambda}^{a_i,n}$ is of order Δa_i with $\Delta a_i = \frac{1}{n}$ and because

$$\frac{1}{n}\sum_{\substack{0\le i\le n-1,\\ a_i<a}}\sqrt{n}\left\|Z_t^{a_i,n}-z_t^{a_i}\right\|_{L^2(P)}\to 0 \qquad n\to\infty$$

by assumption, it follows that

$$\sum_{\substack{0\le i\le n-1,\\ a_i<a}}(Z_t^{a_i,n}-z_t^{a_i})\frac{\beta^n(a_i)\Lambda_t^{a_i,n}}{\eta_{VV}^{a_i,n}}\to 0$$

in $L^1(P)$, and thus in probability as $n\to\infty$, for each fixed $t\ge 0$. On the other hand, Theorem C.9 implies

$$\sum_{\substack{0\le i\le n-1,\\ a_i<a}} z_t^{a_i}\frac{\beta^n(a_i)\Lambda_t^{a_i,n}}{\eta_{VV}^{a_i,n}}\to\left(\frac{\beta(\cdot)}{\eta_{VV}}\cdot\frac{J_x(\cdot,v_0(\cdot),0)}{J_{xx}(\cdot,v_0(\cdot),0)}*\Lambda\right)_{(a,t)},$$

uniformly on compacts in probability, and so by Corollary E.22

$$-\sum_{\substack{0\le i\le n-1,\\ a_i<a}}\frac{J_x^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}{J_{xx}^{a_i,n}(t,V_t^{a_i,n},\Lambda_t^{a_i,n})}\cdot\frac{\beta^n(a_i)\Lambda_t^{a_i,n}}{\eta_{VV}^{a_i,n}}\to\left(\frac{\beta(\cdot)}{\eta_{VV}r\psi(\cdot,rv_0(\cdot))}*\Lambda\right)_{(a,t)}$$

uniformly on compacts in probability, where

$$(\varphi*\Lambda)_{(a,t)}+\alpha_\Lambda\int_0^t(\varphi*\Lambda)_{(a,s)}\,\mathrm{d}s = \int_0^a\varphi(u)(w(u)\mu-\nu(u)\sigma_\xi^2)^2\,\mathrm{d}u\int_0^t\mathrm{d}f_1(D_s,\widehat{G}_s,\Xi_s)$$
$$+\int_{[(0,0),(a,t))}\varphi(u)(\nu(u)\sigma_\xi^2-w(u)\mu)\,\mathrm{d}f_2(\xi_{(u,s)},\widehat{G}_s,\Xi_s)$$

as in Theorem C.9. By defining

$$X_{(a,t)} := \left(\frac{\beta(\cdot)}{\eta_{VV} r\psi(\cdot, rv_0(\cdot))} * \Lambda\right)_{(a,t)} - \int_0^a \frac{\rho}{\eta_{VV}\psi(u, rv_0(u))}\,\mathrm{d}u,$$

it holds that

$$\begin{aligned}
&-\int_0^a \frac{\rho}{\eta_{VV}\psi(u, rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t \big(\tfrac{\beta(\cdot)}{\eta_{VV} r\psi(\cdot, rv_0(\cdot))} * \Lambda\big)_{(a,s)}\,\mathrm{d}s \\
&\quad = -\int_0^a \frac{\rho}{\eta_{VV}\psi(u, rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(a,s)}\,\mathrm{d}s - \alpha_\Lambda \int_0^t \int_0^a \frac{\rho}{\eta_{VV}\psi(u, rv_0(u))}\,\mathrm{d}u\,\mathrm{d}s \\
&\quad = -\int_0^a \frac{(1+\alpha_\Lambda t)\rho}{\eta_{VV}\psi(u, rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(a,s)}\,\mathrm{d}s,
\end{aligned}$$

and this concludes the proof. ■

Lemma F.3. *Let* $\Lambda_t^{a_i,n}$ *and* $\Lambda_t^{a_j,n}$ *be defined as in Corollary C.3. Then* $\mathrm{E}[\Lambda_t^{a_i,n}\Lambda_t^{a_j,n}]$ *is of order* $\max\{(\Delta a_i)^2, (\Delta a_j)^2\}$.

Proof. By using Corollary C.3, applying Itô's formula, and by noting that the private signal at $t=0$ is equal to zero because $\zeta_0^{a_i,n} = \Delta_1 W^{\xi}_{(a_i,0)}$ and the fact that a Brownian sheet is zero whenever one of its coordinate is zero, we find that the inference wedge $\Lambda^{a_i,n}$ satisfies

$$\Lambda_t^{a_i,n} = \int_0^t \mathrm{e}^{-\int_s^t \alpha_\Lambda^n(u)\,\mathrm{d}u} \begin{pmatrix} \Gamma_1^{a_i,n}(s) - \Gamma_1^n(s) \\ \Gamma_2^{a_i,n}(s) - \Gamma_2^n(s) \\ \Gamma_3^{a_i,n}(s) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathrm{d}D_s + \alpha_D(D_s - \widehat{G}_s^{a_i,n})\,\mathrm{d}s \\ \mathrm{d}\Xi_s^n - \mu_n\alpha_G(g - \widehat{G}_s^{a_i,n})\,\mathrm{d}s \\ \mathrm{d}\zeta_s^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G(g - \widehat{G}_s^{a_i,n})\,\mathrm{d}s \end{pmatrix}. \tag{F.1}$$

For our purpose, it is enough to look at the steady state Riccati solution, which essentially means we can work with the constants α_Λ^n, $\Gamma_1^{a_i,n}$, $\Gamma_2^{a_i,n}$, and $\Gamma_3^{a_i,n}$. From Lemma C.6 we know that both differences $\Gamma_1^{a_i,n} - \Gamma_1^n$ and $\Gamma_2^{a_i,n} - \Gamma_2^n$ are of order Δa_i. Moreover, we know from Lemma C.1 that $\Gamma_3^{a_i,n}$ is of order 1, and by definition $\overline{\nu}_{a_i,n} = \nu(a_i)\Delta a_i$ (see Appendix B). We already know that

$$\mathrm{E}[(\Lambda_t^{a_i,n})^2] = \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}(1 - \mathrm{e}^{-2\alpha_\Lambda^n t}) \le \frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n},$$

which together with Hölder's inequality implies

$$\mathrm{E}[|\Lambda_t^{a_i,n}\Lambda_t^{a_j,n}|] \le \mathrm{E}[|\Lambda_t^{a_i,n}|^2]^{1/2}\,\mathrm{E}[|\Lambda_t^{a_j,n}|^2]^{1/2} \le \left(\frac{\eta_{\Lambda\Lambda}^{a_i,n}}{2\alpha_\Lambda^n}\right)^{1/2}\left(\frac{\eta_{\Lambda\Lambda}^{a_j,n}}{2\alpha_\Lambda^n}\right)^{1/2} = \frac{(\eta_{\Lambda\Lambda}^{a_i,n})^{1/2}(\eta_{\Lambda\Lambda}^{a_j,n})^{1/2}}{2\alpha_\Lambda^n}.$$

So $\mathrm{E}[\Lambda_t^{a_i,n}\Lambda_t^{a_j,n}]$ is bounded above by a term that is at least of order $(\Delta a_i)^{1/2}(\Delta a_j)^{1/2}$. To prove the statement we shall show that the mixed terms of order less than $\Delta a_i \Delta a_j$ vanish. It is clear from the representation of $\Lambda_t^{a_i,n}$ above, that all the terms appearing from the expected value $\mathrm{E}[\Lambda_t^{a_i,n}\Lambda_t^{a_j,n}]$ that include a product of the form

$$(\Gamma_k^{a_i,n} - \Gamma_k^n)(\Gamma_l^{a_j,n} - \Gamma_l^n) \qquad k, l \in \{1, 2\}$$

are of order $\Delta a_i \Delta a_j$, by the argumentation above. For the remaining terms, let us first note that $\zeta^{a_i,n}$ is of the form

$$\mathrm{d}\zeta_t^{a_i,n} = \overline{\nu}_{a_i,n}\,\mathrm{d}G_t + m_{a_i,n}\,\mathrm{d}W_t^{\zeta^{a_i,n}} = \nu(a_i)\Delta a_i\,\mathrm{d}G_t + (\Delta a_i)^{1/2}\,\mathrm{d}W_t^{\zeta^{a_i,n}},$$

where $(W_t^{\zeta^{a_i,n}})_{0\le i\le n-1}$ is a collection of independent Brownian motions that is also independent of W^D and W^G. Here, we use that by definition $m_{a_i,n} = (\Delta a_i)^{1/2}$ (see again Appendix B). So the only term in $\Lambda_t^{a_i,n}$ that is not of order Δa_i is the term including $W^{\zeta^{a_i,n}}$, that is

$$\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\Gamma_3^{a_i,n}(\Delta a_i)^{1/2}\,\mathrm{d}W_s^{\zeta^{a_i,n}}.$$

Since W^D and W^G are independent of $W^{\zeta^{a_i,n}}$, so are D and G, which immediately implies

$$\mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\left(\mathrm{d}D_s + \alpha_D D_s\,\mathrm{d}s\right)\right)\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\mathrm{d}W_s^{\zeta^{a_i,n}}\right)\right] = 0.$$

Likewise, since $(W_t^{\zeta^{a_i,n}})_{0\le i\le n-1}$ is a collection of independent Brownian motions it holds that

$$\mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\left(\mathrm{d}\zeta_s^{a_j,n} - \overline{\nu}_{a_j,n}\alpha_G g\,\mathrm{d}s\right)\right)\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,\mathrm{d}W_s^{\zeta^{a_i,n}}\right)\right] = 0 \qquad \forall\, i \neq j.$$

Moreover, because the public signal satisfies

$$\Xi^n = \mu_n G_t + \sum_{i=0}^{n-1} w(a_i) m_{a_i,n} W_t^{\zeta_{a_i},n}$$

we find that

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\left(\mathrm{d}\Xi_s^n - \mu_n\alpha_G g\,\mathrm{d}s\right)\right)\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,m_{a_i,n}\,\mathrm{d}W_s^{\zeta^{a_i,n}}\right)\right] \\
&= \mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\,w(a_i)m_{a_i,n}^2\,\mathrm{d}W_s^{\zeta^{a_i,n}}\right)^2\right] \\
&= w(a_i)^2\Delta a_i \int_0^t \mathrm{e}^{-2\alpha_\Lambda^n(t-s)}\,\mathrm{d}s = \frac{w(a_i)^2\Delta a_i}{2\alpha_\Lambda^n}(1 - \mathrm{e}^{-\alpha_\Lambda^n t}).
\end{aligned}$$

Since Ξ^n is multiplied by $\Gamma_2^{a_i,n} - \Gamma_2^n$ which is of order Δa_i by Lemma C.6 we still obtain

$\Delta a_i \Delta a_j$ here. It thus remains to consider the terms

$$\int_0^t \mathrm{e}^{-\alpha_\Lambda^n (t-s)}\, \widehat{G}_s^{a_i,n}\, \mathrm{d}s$$

in $\Lambda_t^{a_i,n}$. For this purpose, let us first note that Lemma C.1 implies

$$\mathrm{d}\widehat{G}_t^{a_i,n} = (\alpha_G g - (\alpha_\Lambda^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G \Gamma_3^{a_i,n})\widehat{G}_t^{a_i,n})\, \mathrm{d}t + \begin{pmatrix} \Gamma_1^{a_i,n} \\ \Gamma_2^{a_i,n} \\ \Gamma_3^{a_i,n} \end{pmatrix}^\top \begin{pmatrix} \mathrm{d}D_t + \alpha_D D_t\, \mathrm{d}t \\ \mathrm{d}\Xi_t^n - \mu_n \alpha_G g\, \mathrm{d}t \\ \mathrm{d}\zeta_t^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G g\, \mathrm{d}t \end{pmatrix},$$

where we denote $\alpha_\Lambda^{a_i,n} \coloneqq \alpha_D \Gamma_1^{a_i,n} + \alpha_G(1 - \mu_n \Gamma_2^{a_i,n})$. By Itô's formula we then have

$$\begin{aligned} \widehat{G}_t^{a_i,n} = {} & \widehat{G}_0^{a_i,n}\, \mathrm{e}^{-(\alpha_\Lambda^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G\Gamma_3^{a_i,n})t} + \alpha_G g \int_0^t \mathrm{e}^{-(\alpha_\Lambda^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G\Gamma_3^{a_i,n})(t-s)}\, \mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-(\alpha_\Lambda^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G\Gamma_3^{a_i,n})(t-s)} \begin{pmatrix} \Gamma_1^{a_i,n} \\ \Gamma_2^{a_i,n} \\ \Gamma_3^{a_i,n} \end{pmatrix}^\top \begin{pmatrix} \mathrm{d}D_s + \alpha_D D_s\, \mathrm{d}s \\ \mathrm{d}\Xi_s^n - \mu_n \alpha_G g\, \mathrm{d}s \\ \mathrm{d}\zeta_s^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G g\, \mathrm{d}s \end{pmatrix}. \end{aligned}$$

Hence, by the exact same argumentation as above, we find that the only term contributing anything here is Ξ^n, so that

$$\begin{aligned} & \mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\, \widehat{G}_s^{a_j,n}\, \mathrm{d}s\right)\left(\int_0^t \mathrm{e}^{-\alpha_\Lambda^n(t-s)}\, m_{a_i,n}\, \mathrm{d}W_s^{\zeta^{a_i,n}}\right)\right] \\ & = \frac{\Gamma_2^{a_i,n} w(a_i)\Delta a_i}{\mathrm{e}^{\alpha_\Lambda^n t}}\, \mathrm{E}\left[\int_0^t \mathrm{e}^{\alpha_\Lambda^n s}\left(\int_0^s \mathrm{e}^{-(\alpha_\Lambda^{a_i,n} - \overline{\nu}_{a_i,n}\alpha_G\Gamma_3^{a_i,n})(s-u)}\, \mathrm{d}W_u^{\zeta^{a_i,n}}\right) \mathrm{d}s \left(\int_0^t \mathrm{e}^{\alpha_\Lambda^n s}\, \mathrm{d}W_s^{\zeta^{a_i,n}}\right)\right]. \end{aligned}$$

Note that here Ξ^n does not enter with an additional factor $\Gamma_2^{a_i,n} - \Gamma_2^n$ as above. However, it can be seen from (F.1) that each term $\widehat{G}^{a_i,n}\, \mathrm{d}t$ is multiplied by a factor Δa_i, either via $\Gamma_1^{a_i,n} - \Gamma_1^n$, $\Gamma_2^{a_i,n} - \Gamma_2^n$, or $\overline{\nu}_{a_i,n}$. This is enough to conclude the proof. ■

F.2. Risk Premium

Proposition F.4. *The risk premium ρ satisfies*

$$\rho = -\frac{\eta_{VV}\theta}{\int_0^1 \frac{1}{\psi(a, r v_0(a))}\, \mathrm{d}a}.$$

Proof. Let us assume that we have found a weight function w which reduces the cumulative demand function $X_{(1,t)}$ to (MC′). Our goal is to derive the risk premium ρ. For this purpose, we substitute (MC) into (MC′), which yields

$$-r\rho(1+\alpha_\Lambda t) = \frac{r\eta_{VV}\theta(1+\alpha_\Lambda t)}{\int_0^1 \frac{1}{\psi(a,rv_0(a))}\,\mathrm{d}a}. \tag{F.2}$$

Solving equation (F.2) for ρ yields

$$\rho = -\frac{\eta_{VV}\theta}{\int_0^1 \frac{1}{\psi(a,rv_0(a))}\,\mathrm{d}a},$$

and this concludes the proof. ■

F.3. Equilibrium conditions

We first give the proof of Proposition 3.1 which gives a necessary and sufficient condition for a trivial equilibrium to exist.

Proof of Proposition 3.1. Let us assume that $w\colon [0,1) \to \mathbb{R}$ is a weight function that yields a trivial equilibrium. By definition and from the cumulative demand function (3.3) it must at least hold true that

$$\int_0^a \frac{\beta(u)(w(u)\mu - \nu(u)\sigma_\xi^2)^2}{r\eta_{VV}\psi(u,rv_0(u))}\,\mathrm{d}u = 0 \qquad \forall a \in [0,1).$$

Hence, by the Lebesgue differentiation theorem, it must hold $\frac{\beta(a)(w(a)\mu-\nu(a)\sigma_\xi^2)^2}{r\eta_{VV}\psi(a,rv_0(a))} = 0$ for any $a \in \{\beta \neq 0\}$, and therefore $w(a)\mu = \nu(a)\sigma_\xi^2$. This implies $w(a) = \nu(a)\sigma_\xi^2/\mu$ for all $a \in \{\beta \neq 0\}$. To prove the reverse implication, let us assume that $w(a) = \nu(a)\sigma_\xi^2/\mu$ for all $a \in \{\beta \neq 0\}$. Hence, equation (3.3) reduces to

$$X_{(a,t)} = -\frac{\rho\,(1+\alpha_\Lambda t)}{\eta_{VV}}\int_0^a \frac{1}{\psi(u,rv_0(u))}\,\mathrm{d}u - \alpha_\Lambda \int_0^t X_{(a,s)}\,\mathrm{d}s$$

for almost every $a \in [0,1)$, which by definition guarantees the existence of a trivial equilibrium. ■

Next, we give the proof of Proposition 3.2 which provides a characterization of the weight function in a non-trivial equilibrium when it exists.

Proof of Proposition 3.2. In order to prove the statement, we prove the two implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2): Let us assume that there exists a non-trivial equilibrium, which means that there exists a well-defined weight function w that clears the market, which is not proportional to ν

on $\{\beta \neq 0\}$. Furthermore, it means that $\{\beta \neq 0\}$ has positive Lebesgue measure. According to equation (3.3), in order to have market clearing, the weight function must at least satisfy

$$\frac{\beta(a)(\mu w(a) - \sigma_\xi^2 \nu(a))^2}{\psi(a, r v_0(a))} = \tilde{f}(a) \quad \text{for a.e. } a \in [0,1), \tag{F.3}$$

for some non-zero function $\tilde{f}\colon [0,1) \to \mathbb{R}$ satisfying $\int_0^1 \tilde{f}(a)\,\mathrm{d}a = 0$. The function $\tilde{f}$ must be non-zero in order to obtain a non-trivial equilibrium, while the condition $\int_0^1 \tilde{f}(a)\,\mathrm{d}a = 0$ is required for the market to clear. A particular candidate for such a function is given by $\tilde{f}(a) = \kappa w(a)(\mu w(a) - \sigma_\xi^2 \nu(a))$ for a.e. $a \in [0,1)$, for some coefficient $\kappa \neq 0$. In fact, this particular choice of $\tilde{f}$ is the only one that clears the market in a non-trivial way. Indeed, let us denote

$$\int_0^1 \frac{\beta(u)(\mu \sigma_G^2 \nu(u) + w(u))(\mu w(u) - \sigma_\xi^2 \nu(u))}{\psi(u, r v_0(u))(\mu^2 \sigma_G^2 + \sigma_\xi^2)}\,\mathrm{d}u =: \kappa_0.$$

Then, in order to have market clearing, equation (3.3) implies (see the third and the fifth terms on the right-hand side of (3.3))

$$\int_{[(0,0),(1,t))} \left(\kappa_0 w(u) - \frac{\beta(u)(\mu w(u) - \nu(u)\sigma_\xi^2)}{\psi(u, r v_0(u))} \right) \mathrm{d}W^\xi_{(u,s)} = 0 \quad P\text{-a.s.},$$

and thus by the Itô isometry for Brownian sheets we obtain

$$\begin{aligned} &\mathrm{E}\left[\left(\int_{[(0,0),(1,t))} \left(\kappa_0 w(u) - \frac{\beta(u)(\mu w(u) - \nu(u)\sigma_\xi^2)}{\psi(u, r v_0(u))} \right) \mathrm{d}W^\xi_{(u,s)} \right)^2 \right] \\ &\qquad = t \int_0^1 \left(\kappa_0 w(u) - \frac{\beta(u)(\mu w(u) - \nu(u)\sigma_\xi^2)}{\psi(u, r v_0(u))} \right)^2 \mathrm{d}u = 0, \end{aligned}$$

which holds true if and only if

$$\kappa_0 w(a) - \frac{\beta(a)(\mu w(a) - \sigma_\xi^2 \nu(a))}{\psi(a, r v_0(a))} = 0 \quad \text{for a.e. } a \in [0,1). \tag{F.4}$$

Hence, we find from (F.3) and (F.4) that

$$\frac{\beta(a)(\mu w(a) - \sigma_\xi^2 \nu(a))}{\psi(a, r v_0(a))} = \frac{\tilde{f}(a)}{\mu w(a) - \sigma_\xi^2 \nu(a)} = \kappa_0 w(a),$$

which implies that $\tilde{f}(a) = \kappa_0 w(a)(\mu w(a) - \nu(a)\sigma_\xi^2)$. By letting

$$\frac{\beta(a)(\mu w(a) - \sigma_\xi^2 \nu(a))}{\psi(a, r v_0(a))} = \kappa w(a),$$

for some general $\kappa \neq 0$, we get

$$\begin{aligned}\kappa_0 &= \int_0^1 \frac{\beta(u)(\mu\sigma_G^2\nu(u) + w(u))(\mu w(u) - \sigma_\xi^2\nu(u))}{\psi(u, rv_0(u))(\mu^2\sigma_G^2 + \sigma_\xi^2)}\,\mathrm{d}u \\ &= \kappa \int_0^1 \frac{\mu\sigma_G^2 w(u)\nu(u) + w(u)^2}{\mu^2\sigma_G^2 + \sigma_\xi^2}\,\mathrm{d}u = \kappa.\end{aligned}$$

As a result, we obtain that the weight function must be of the general form

$$w(a) = \frac{\sigma_\xi^2\nu(a)}{\mu + \kappa\frac{\psi(a, rv_0(a))}{\beta(a)}} = \frac{\sigma_\xi^2\nu(a)\beta(a)}{\mu\beta(a) + \kappa\psi(a, rv_0(a))} \quad \text{for a.e. } a \in [0, 1), \qquad \kappa \neq 0. \tag{F.5}$$

From (F.5) we observe immediately that $w(a) = 0$ for all $a \in \{\beta = 0\}$.

(2) $\Longrightarrow$ (1): Let us now assume that the weight function is well-defined and of the form (F.5), and that the set $\{\beta \neq 0\}$ has positive Lebesgue measure. We thus obtain that

$$\frac{\beta(a)(\mu w(a) - \sigma_\xi^2\nu(a))}{\psi(a, rv_0(a))} = -\kappa w(a) \qquad \text{for a.e. } a \in [0, 1),$$

and therefore

$$\int_0^1 \frac{\beta(u)(\mu w(u) - \sigma_\xi^2\nu(u))^2}{\psi(u, rv_0(u))}\,\mathrm{d}u = \kappa \int_0^1 (\sigma_\xi^2\nu(u)w(u) - \mu w(u)^2)\,\mathrm{d}u = 0.$$

Moreover, we find that

$$\begin{aligned}&\left(\int_0^1 \frac{\beta(u)(\mu\sigma_G^2\nu(u) + w(u))(\mu w(u) - \sigma_\xi^2\nu(u))}{\psi(u, rv_0(u))(\mu^2\sigma_G^2 + \sigma_\xi^2)}\,\mathrm{d}u\right)\big(\Xi_t - \mu\alpha_G\textstyle\int_0^t(g - \widehat{G}_s)\,\mathrm{d}s\big) \\ &+ \int_{[(0,0),(1,t))} \frac{\beta(u)(\nu(u)\sigma_\xi^2 - \mu w(u))}{\psi(u, rv_0(u))}(\mathrm{d}\xi_{(u,s)} - \nu(u)\alpha_G(g - \widehat{G}_s)\,\mathrm{d}u\,\mathrm{d}s) \\ &= \kappa\big(\Xi_t - \mu\alpha_G\textstyle\int_0^t(g - \widehat{G}_s)\,\mathrm{d}s\big) - \kappa\big(\Xi_t - \mu\alpha_G\textstyle\int_0^t(g - \widehat{G}_s)\,\mathrm{d}s\big) = 0,\end{aligned}$$

which implies

$$X_{(1,t)} = -\frac{\rho\,(1 + \alpha_\Lambda t)}{\eta_{VV}} \int_0^1 \frac{1}{\psi(a, rv_0(a))}\,\mathrm{d}a - \alpha_\Lambda \int_0^t X_{(1,s)}\,\mathrm{d}s \qquad P\text{-a.s.},$$

and so the market clears under the condition (MC′). ■

F.4. Existence of non-trivial equilibria

We first prove Proposition 3.4 which gives a necessary and sufficient condition for a non-trivial equilibrium to exist.

Proof of Proposition 3.4. If (3.6) holds, set $w(a) = \frac{\nu(a)}{1+\tilde{\kappa}\frac{\psi(a,rv_0(a))}{\beta(a)}} \mathbb{1}_{\{\beta \neq 0\}}(a)$. We find that

$$\begin{aligned}
\sigma_\xi^2 &= \int_0^1 w(a)^2 \,\mathrm{d}a = \int_{\{\beta\neq 0\}} \frac{\nu(a)^2}{\left(1+\tilde{\kappa}\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \,\mathrm{d}a \\
&= \int_{\{\beta\neq 0\}} \frac{\nu(a)^2}{\left(1+\tilde{\kappa}\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \,\mathrm{d}a + \int_{\{\beta\neq 0\}} \frac{\nu(a)^2}{\left(1+\tilde{\kappa}\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \cdot \frac{\tilde{\kappa}\psi(a,rv_0(a))}{\beta(a)} \,\mathrm{d}a \\
&= \int_{\{\beta\neq 0\}} \frac{\nu(a)^2}{1+\tilde{\kappa}\frac{\psi(a,rv_0(a))}{\beta(a)}} \,\mathrm{d}a = \int_0^1 \nu(a) w(a) \,\mathrm{d}a = \mu.
\end{aligned}$$

The condition that $\{\beta \neq 0\}$ has positive measure guarantees that $\int_0^1 w(a)^2 \,\mathrm{d}a > 0$ and thus $\sigma_\xi^2 = \mu > 0$. Therefore, we can write

$$w(a) = \frac{\nu(a)}{1+\tilde{\kappa}\frac{\psi(a,rv_0(a))}{\beta(a)}} \mathbb{1}_{\{\beta\neq 0\}}(a) = \frac{\sigma_\xi^2 \nu(a)}{\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}} \mathbb{1}_{\{\beta\neq 0\}}(a)$$

for $\kappa = \mu\tilde{\kappa} \neq 0$, which shows that a non-trivial equilibrium exists by Proposition 3.2.

Conversely, Proposition 3.2 implies that the weight function is of the form

$$w(a) = \frac{\sigma_\xi^2 \nu(a)}{\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}} \mathbb{1}_{\{\beta\neq 0\}}(a)$$

or some $\kappa \neq 0$ and that $\{\beta \neq 0\}$ has positive measure. Hence, we find that

$$\sigma_\xi^2 = \int_0^1 w(a)^2 \,\mathrm{d}a = \int_{\{\beta\neq 0\}} \frac{\sigma_\xi^4 \nu(a)^2}{\left(\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \,\mathrm{d}a,$$

which implies

$$\int_{\{\beta\neq 0\}} \frac{\sigma_\xi^2 \nu(a)^2}{\left(\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \,\mathrm{d}a = 1. \tag{F.6}$$

Moreover, we have

$$\begin{aligned}
\mu &= \int_0^1 \nu(a) w(a) \,\mathrm{d}a = \int_{\{\beta\neq 0\}} \frac{\sigma_\xi^2 \nu(a)^2}{\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}} \,\mathrm{d}a \\
&= \mu \int_{\{\beta\neq 0\}} \frac{\sigma_\xi^2 \nu(a)^2}{\left(\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \,\mathrm{d}a + \int_{\{\beta\neq 0\}} \frac{\sigma_\xi^2 \nu(a)^2}{\left(\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2} \kappa \frac{\psi(a,rv_0(a))}{\beta(a)} \,\mathrm{d}a,
\end{aligned}$$

which together with (F.6), yields

$$\int_{\{\beta\neq 0\}} \frac{\sigma_\xi^2 \nu(a)^2}{\left(\mu+\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2}\kappa\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a=0,$$

and thus

$$\int_{\{\beta\neq 0\}} \frac{\nu(a)^2}{\left(1+\frac{\kappa}{\mu}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a=0.$$

This is of the form (3.6) with $\tilde{\kappa}=\kappa/\mu$. ■

Next we prove the existence result of non-trivial equilibria given in Proposition 3.5.

Proof of Proposition 3.5. Consider the functions

$$\begin{aligned}
I(\tilde{\kappa}) &= \int_{\{\beta>0\}} \frac{\nu(a)^2}{\left(1+\frac{\tilde{\kappa}\psi(a,rv_0(a))}{\beta(a)}\right)^2}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a,\\
J(\tilde{\kappa}) &= -\int_{\{\beta<0\}} \frac{\nu(a)^2}{\left(1+\frac{\tilde{\kappa}\psi(a,rv_0(a))}{\beta(a)}\right)^2}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a.
\end{aligned}$$

From (3.7), we know that $I(0)\neq J(0)$.

First assume that $I(0)>J(0)$. We note that $J(\tilde{\kappa})$ is continuous on $[0,s_1)$, where s_1 is the first blow-up time to infinity. We next show that such blow-ups indeed happen. Fix $a_1\in(0,1)$ with $\beta(a_1)<0$. Because $\beta(a)$ and $\psi(a,rv_0(a))$ are locally Lipschitz-continuous on $\{\beta\neq 0\}$, $\psi(a,rv_0(a))/\beta(a)$ is locally Lipschitz-continuous at a_1; denote by L_1 the related Lipschitz constant. We have

$$\begin{aligned}
J\left(\frac{-\beta(a_1)}{\psi(a_1,rv_0(a_1))}\right) &= -\int_{\{\beta<0\}} \frac{\nu(a)^2}{\left(1-\frac{\beta(a_1)\psi(a,rv_0(a))}{\psi(a_1,rv_0(a_1))\beta(a)}\right)^2}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a\\
&= -\int_{\{\beta<0\}} \frac{\psi(a_1,rv_0(a_1))^2\nu(a)^2}{\beta(a_1)^2\left(\frac{\psi(a_1,v_0(a_1))}{\beta(a_1)}-\frac{\psi(a,rv_0(a))}{\beta(a)}\right)^2}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a\\
&\geq -\int_{a_1-\epsilon}^{a_1} \frac{\psi(a_1,rv_0(a_1))^2\nu(a)^2}{\beta(a_1)^2L_1^2(a_1-a)^2}\cdot\frac{\psi(a,rv_0(a))}{\beta(a)}\,\mathrm{d}a
\end{aligned}$$

for a suitably chosen $\epsilon>0$. Because the integral of $(a_1-a)^{-2}$ over $(a_1-\epsilon,a_1)$ diverges, we obtain $J\left(\frac{-\beta(a_1)}{\psi(a_1,rv_0(a_1))}\right)=\infty$. Moreover, the assumption $\sup_{\{\beta<0\}}\beta<0$ guarantees that the blow-up does not happen immediately at zero, so $s_1>0$. Since $I(0)>J(0)$, $I(\tilde{\kappa})$ is continuous and decreasing on $[0,\infty)$, and $J(\tilde{\kappa})$ is continuous on $[0,s_1)$ and diverges to ∞ at s_1, there exists $\tilde{\kappa}^\star\in(0,s_1)$ such that $I(\tilde{\kappa}^\star)=J(\tilde{\kappa}^\star)$.

Now assume that $I(0)<J(0)$. We note that $I(\tilde{\kappa})$ is continuous on $(s_2,0]$, where s_2 is the last blow-up time to infinity before zero. We next show that such blow-ups indeed

happen. Fix $a_2 \in (0,1)$ with $\beta(a_2) > 0$. Because $\beta(a)$ and $\psi(a, rv_0(a))$ are locally Lipschitz-continuous on $\{\beta \neq 0\}$, $\psi(a, rv_0(a))/\beta(a)$ is locally Lipschitz-continuous at a_2; denote by L_2 the related Lipschitz constant. We have

$$\begin{aligned} I\bigg(\frac{-\beta(a_2)}{\psi(a_2, rv_0(a_2))}\bigg) &= \int_{\{\beta>0\}} \frac{\nu(a)^2}{\big(1 - \frac{\beta(a_2)\psi(a,rv_0(a))}{\psi(a_2,rv_0(a_2))\beta(a)}\big)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)} \,\mathrm{d}a \\ &= \int_{\{\beta>0\}} \frac{\psi(a_2, rv_0(a_2))^2\nu(a)^2}{\beta(a_2)^2\big(\frac{\psi(a_2,rv_0(a_2))}{\beta(a_2)} - \frac{\psi(a,rv_0(a))}{\beta(a)}\big)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)} \,\mathrm{d}a \\ &\geq \int_{a_2}^{a_2+\epsilon} \frac{\psi(a_2, rv_0(a_2))^2\nu(a)^2}{\beta(a_2)^2 L_2^2 (a_2 - a)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)} \,\mathrm{d}a \end{aligned}$$

for a suitably chosen $\epsilon > 0$. Because the integral of $(a_2 - a)^{-2}$ over $(a_2, a_2 + \epsilon)$ diverges, we obtain $I\big(\frac{-\beta(a_2)}{\psi(a_2,rv_0(a_2))}\big) = \infty$. Moreover, the assumption $\inf_{\{\beta>0\}} \beta > 0$ guarantees that the blow-up does not happen immediately at zero, so $s_2 < 0$. Since $I(0) < J(0)$, $J(\tilde{\kappa})$ is continuous and increasing on $(-\infty, 0]$, and $I(\tilde{\kappa})$ is continuous on $(s_2, 0]$ and diverges to ∞ at s_2, there exists $\tilde{\kappa}^\star \in (s_2, 0)$ such that $I(\tilde{\kappa}^\star) = J(\tilde{\kappa}^\star)$.

In either case, we obtain

$$\int_{\{\beta\neq 0\}} \frac{\nu(a)^2}{\big(1 + \tilde{\kappa}^\star \frac{\psi(a,rv_0(a))}{\beta(a)}\big)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)} \,\mathrm{d}a = 0$$

for $\tilde{\kappa}^\star \neq 0$, which shows that a non-trivial equilibrium exists by Proposition 3.4. ∎

G. Proofs of Sections 4 and 5

This appendix contains the proofs of the results and statements from Sections 4 and 5. Appendix G.1 covers Section 4, while Appendix G.2 covers Section 5.

G.1. Key financial market measures

Proof of Proposition 4.1. The conditional variance is defined as $\mathrm{Var}[G_t|\mathcal{G}_t] = \mathrm{E}[(G_t - \widehat{G}_t)^2|\mathcal{G}_t]$, which implies $\mathrm{E}[\mathrm{Var}[G_t|\mathcal{G}_t]] = \mathrm{E}[(G_t - \widehat{G}_t)^2] = \gamma_t$. This is the mean squared filtering error described by the Riccati equation (C.7). On the other hand, to compute the variance of G_t, recall that $\mathrm{d}G_t = \alpha_G(g - G_t)\,\mathrm{d}t + \sigma_G\,\mathrm{d}W_t^G$, and so we obtain by Itô's lemma that

$$G_t = \mathrm{e}^{-\alpha_G t}\,G_0 + g(1 - \mathrm{e}^{-\alpha_G t}) + \sigma_G \int_0^t \mathrm{e}^{\alpha_G(s-t)}\,\mathrm{d}W_s^G,$$

which implies $\mathrm{Var}(G_t) = \frac{1-\mathrm{e}^{-2\alpha_G t}}{2\alpha_G}\sigma_G^2$. Hence, we find that price informativeness satisfies

$$\mathcal{I} = 1 - \lim_{t\to\infty} \frac{\mathrm{E}[\mathrm{Var}[G_t|\mathcal{G}_t]]}{\mathrm{Var}(G_t)} = 1 - \lim_{t\to\infty} \frac{2\alpha_G\gamma_t}{\sigma_G^2(1-\mathrm{e}^{-2\alpha_G t})} = 1 - \frac{2\alpha_G\gamma}{\sigma_G^2}.$$

Since the steady state mean squared filtering error satisfies

$$\gamma = \frac{\sigma_D\sqrt{(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)(\tau^2\sigma_G^2+1)} - \alpha_G\sigma_D^2}{\tau^2(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)+\alpha_D^2}, \qquad \tau^2 := \frac{\mu^2}{\sigma_\xi^2},$$

we thus obtain

$$\begin{aligned}
\mathcal{I} &= 1 - \frac{2\alpha_G}{\sigma_G^2}\cdot\frac{\sigma_D\sqrt{(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)(\tau^2\sigma_G^2+1)} - \alpha_G\sigma_D^2}{\tau^2(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)+\alpha_D^2} \\
&= \frac{(\tau^2\sigma_G^2+1)(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2) - 2\sigma_D\alpha_G\sqrt{(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)(\tau^2\sigma_G^2+1)} + \alpha_G^2\sigma_D^2}{\tau^2\sigma_G^2(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)+\alpha_D^2\sigma_G^2} \\
&= \frac{\left(\sqrt{(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)(\tau^2\sigma_G^2+1)} - \sigma_D\alpha_G\right)^2}{\tau^2\sigma_G^2(\alpha_D^2\sigma_G^2+\alpha_G^2\sigma_D^2)+\alpha_D^2\sigma_G^2}.
\end{aligned}$$

This concludes the proof. ∎

Proof of Proposition 4.2. From the market clearing equation (3.3) we immediately get

$$\begin{aligned}
\langle X, X\rangle_{(a,t)} &= \frac{(\sigma_G^2-\alpha_G\gamma)^2}{(\mu^2\sigma_G^2+\sigma_\xi^2)^2}\int_{[(0,0),(a,t))} \frac{\beta(u)^2(\nu(u)\sigma_\xi^2 - w(u)\mu)^2}{r^2\eta_{VV}^2\psi(u,rv_0(u))^2}\,\mathrm{d}u\,\mathrm{d}s \\
&= \frac{(\sigma_G^2-\alpha_G\gamma)^2}{(\mu^2\sigma_G^2+\sigma_\xi^2)^2} t\int_0^a \frac{\beta(u)^2(\nu(u)\sigma_\xi^2 - w(u)\mu)^2}{r^2\eta_{VV}^2\psi(u,rv_0(u))^2}\,\mathrm{d}u,
\end{aligned}$$

which implies

$$\frac{\partial\langle X, X\rangle_{(a,t)}}{\partial t} = \frac{(\sigma_G^2-\alpha_G\gamma)^2}{(\mu^2\sigma_G^2+\sigma_\xi^2)^2}\int_0^a \frac{\beta(u)^2(\nu(u)\sigma_\xi^2 - w(u)\mu)^2}{r^2\eta_{VV}^2\psi(u,rv_0(u))^2}\,\mathrm{d}u.$$

Taking the square root concludes the proof. ∎

Proof of Proposition 4.4. Recall from Proposition 3.1 and Proposition 3.2 that the weight function w that describes the equilibrium can be written as

$$w(a) = \frac{\sigma_\xi^2\nu(a)}{\mu + \kappa\frac{\psi(a,rv_0(a))}{\beta(a)}}\mathbb{1}_{\{\beta\neq 0\}}(a), \qquad \kappa\in\mathbb{R}.$$

Note that $\kappa\in\mathbb{R}$, since we want to describe both, trivial and non-trivial equilibria. Solving for κ and then integrating over a from 0 to 1 thus yields

$$\kappa = \int_0^1 \left(\frac{\sigma_\xi^2 \nu(a)}{w(a)} - \mu \right) \frac{\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a = \int_{\{\beta \neq 0\}} \left(\frac{\sigma_\xi^2 \nu(a)}{w(a)} - \mu \right) \frac{\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a.$$

Since $\mu \neq 0$ by assumption, we may divide both sides of the above equation by μ, and thus obtain

$$\frac{\kappa}{\mu} = \int_{\{\beta \neq 0\}} \left(\frac{\sigma_\xi^2 \nu(a)}{\mu w(a)} - 1 \right) \frac{\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a.$$

By denoting $\tilde{\kappa} = \frac{\kappa}{\mu}$, and by realizing that the weight function in a trivial equilibrium is of the form $w_{\text{trivial}}(a) = \frac{\sigma_\xi^2 \nu(a)}{\mu}$ for any $a \in \{\beta \neq 0\}$ (see Proposition 3.1), we may thus write

$$\tilde{\kappa} = \int_{\{\beta \neq 0\}} \left(\frac{w_{\text{trivial}}(a)}{w(a)} - 1 \right) \frac{\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a.$$

Therefore, we see that the level of disagreement in the market satisfies

$$|\tilde{\kappa}| = \left| \int_{\{\beta \neq 0\}} \left(\frac{w_{\text{trivial}}(a)}{w(a)} - 1 \right) \frac{\beta(a)}{\psi(a, rv_0(a))} \, \mathrm{d}a \right| = \int_{\{\beta \neq 0\}} \left| \frac{w_{\text{trivial}}(a)}{w(a)} - 1 \right| \frac{|\beta(a)|}{\psi(a, rv_0(a))} \, \mathrm{d}a.$$

The second identity follows from

$$\tilde{\kappa} = \left(\frac{w_{\text{trivial}}(a)}{w(a)} - 1 \right) \frac{\beta(a)}{\psi(a, rv_0(a))} \qquad \forall a \in \{\beta \neq 0\}.$$

This concludes the proof. ■

Proof of Proposition 4.5. By assumption, there exists a non-trivial equilibrium for the belief-function β. Hence, by Proposition 3.4 it holds that

$$\int_{\{\beta \neq 0\}} \frac{\nu(a)^2}{\left(1 + \tilde{\kappa} \frac{\psi(a, rv_0(a))}{\beta(a)}\right)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a = 0$$

for some $\tilde{\kappa} \neq 0$. However, for any $c \neq 0$ it holds that

$$\int_{\{\beta' \neq 0\}} \frac{\nu(a)^2}{\left(1 + \tilde{\kappa}' \frac{\psi(a, rv_0(a))}{\beta'(a)}\right)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta'(a)} \, \mathrm{d}a = \frac{1}{c} \int_{\{\beta \neq 0\}} \frac{\nu(a)^2}{\left(1 + \frac{\tilde{\kappa}'}{c} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)}\right)^2} \cdot \frac{\psi(a, rv_0(a))}{\beta(a)} \, \mathrm{d}a,$$

which is equal to 0 if $\frac{\tilde{\kappa}'}{c} = \tilde{\kappa}$, or equivalently $\tilde{\kappa}' = c\tilde{\kappa}$. Hence, by Proposition 3.4 a non-trivial equilibrium exists for the belief function $\beta' = c\beta$ with $\tilde{\kappa}' = c\tilde{\kappa}$. Its weight function satisfies

$$w'(a) = \frac{(\sigma_\xi')^2 \nu(a)}{\mu' + \kappa' \frac{\psi(a, rv_0(a))}{\beta'(a)}} \mathbb{1}_{\{\beta' \neq 0\}}(a) = \frac{(\sigma_\xi')^2}{\mu'} \cdot \frac{\nu(a)}{1 + \frac{\kappa'}{\mu'} \cdot \frac{\psi(a, rv_0(a))}{\beta'(a)}} \mathbb{1}_{\{\beta' \neq 0\}}(a) = w(a),$$

where we use that $\frac{(\sigma'_\xi)^2}{\mu'} = 1$, $\frac{\kappa'}{\mu'} = \tilde{\kappa}' = c\tilde{\kappa}$ and $\{\beta' \neq 0\} = \{\beta \neq 0\}$. Hence, the weight function remains the same. ■

Proof of Corollary 4.6. This statement follows immediately from Proposition 4.5 by letting $\beta' = \frac{1}{c}\beta$, and keeping ψ as is. ■

G.2. Market quality, wealth effects and the role of beliefs

Proof of Proposition 5.1. To prove (1), we note that

$$\|w_L\|_1 = \int_{\{\beta>0\}} \frac{\nu(a)}{\left|1 - |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}\right|} \,\mathrm{d}a \geq \int_{\{\beta>0\}} \frac{\nu(a)}{|\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} \,\mathrm{d}a,$$

as well as

$$\|w_H\|_1 = \int_{\{\beta<0\}} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} \,\mathrm{d}a \leq \int_{\{\beta<0\}} \frac{\nu(a)}{|\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} \,\mathrm{d}a.$$

Since $\tilde{\kappa} < 0$, the strict inequality in (5.3) holds true, which by assumption implies

$$\int_{\{\beta>0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \,\mathrm{d}a > -\int_{\{\beta<0\}} \frac{\nu(a)\beta(a)}{\psi(a, rv_0(a))} \,\mathrm{d}a,$$

and therefore $\|w_L\|_1 > \|w_H\|_1$.

Assertion (2) follows from (1) by replacing H with L and vice versa. ■

Proof of Proposition 5.2. Let us first assume that $v_0(a) \geq v_0(a')$ implies $\nu(a) \geq \nu(a')$ and $|\beta(a)| \geq |\beta(a')|$ for all $a, a' \in [0, 1)$. Since in addition $v_0(a) > v_0(a')$ implies $\psi(a, rv_0(a)) < \psi(a', rv_0(a'))$ for all $a, a' \in [0, 1)$, we find that

$$\frac{\nu(a)}{\left|1 - |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}\right|} \geq \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} > \frac{\nu(a')}{1 + |\tilde{\kappa}| \frac{\psi(a', rv_0(a'))}{|\beta(a')|}} \tag{G.1}$$

for all $\tilde{\kappa} \in \mathbb{R}$. Now, if $L^+(\beta) = \emptyset$ and $\tilde{\kappa} < 0$ or $L^-(\beta) = \emptyset$ and $\tilde{\kappa} > 0$, then the weight function of the less wealthy agents satisfies

$$w_L(a) = \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} \mathbb{1}_L(a), \tag{G.2}$$

and it immediately follows from (G.1) that $\|w_H\|_1 > \|w_L\|_1$.

Now, let us assume that $v_0(a) \geq v_0(a')$ implies $|\beta(a)| \leq |\beta(a')|$ for all $a, a' \in [0, 1)$. Moreover, assume that

$$\min_{a \in L(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}} > \max_{a \in H(\beta)} \frac{\nu(a)}{1 + |\tilde{\kappa}| \frac{\psi(a, rv_0(a))}{|\beta(a)|}}.$$

Hence, the inequalities in (G.1) still hold true but with $a, a' \in [0,1)$ such that $v_0(a') > v_0(a)$. If $H^+(\beta) = \emptyset$ and $\tilde{\kappa} < 0$ or $H^-(\beta) = \emptyset$ and $\tilde{\kappa} > 0$, then the weight function of the wealthier agents is of the form given in equation (G.2), which now implies $\|w_L\|_1 > \|w_H\|_1$. ■

Proof of Proposition 5.5. Under the assumptions of assertion (1a), total precision of the initial wealth allocation is given by

$$\mu^{\text{old}} = \int_{L(\beta)} \frac{\nu(a)^2}{1 - |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}} \, \mathrm{d}a + \int_{H(\beta)} \frac{\nu(a)^2}{1 + |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}} \, \mathrm{d}a,$$

and total precision of the new wealth allocation is given by

$$\mu^{\text{new}} = \int_{L(\beta)} \frac{\nu(a)^2}{1 - |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|}} \, \mathrm{d}a + \int_{H(\beta)} \frac{\nu(a)^2}{1 + |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|}} \, \mathrm{d}a.$$

Since all investors have DARA utilities we find that $\psi(a, rv_0^{\text{new}}(a)) \leq \psi(a, rv_0^{\text{old}}(a))$ for all $a \in L(\beta)$, whereas $\psi(a, rv_0^{\text{new}}(a)) \geq \psi(a, rv_0^{\text{old}}(a))$ for all $a \in H(\beta)$. Hence, if $0 > \tilde{\kappa}^{\text{new}} \geq \tilde{\kappa}^{\text{old}}$, it holds that $|\tilde{\kappa}^{\text{new}}| \leq |\tilde{\kappa}^{\text{old}}|$, and thus by assumption we find for all $a \in L(\beta)$ that

$$1 - |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \geq 1 - |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \geq 1 - \max_{a \in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} > 0,$$

and therefore $0 < w_L^{\text{new}}(a) \leq w_L^{\text{old}}(a)$ for all $a \in L(\beta)$. On the other hand, for all $a \in H(\beta)$ it holds by assumption that

$$\begin{aligned}
1 + |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} &\leq 1 + \max_{a \in H(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \\
&\leq 1 + \min_{a \in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \leq 1 + |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|},
\end{aligned}$$

and therefore $0 < w_H^{\text{new}}(a) \leq w_H^{\text{old}}(a)$ for all $a \in H(\beta)$. Consequently it holds that $\mu^{\text{old}} \geq \mu^{\text{new}}$, and thus by Proposition 4.1 it follows that $\mathcal{I}^{\text{old}} \geq \mathcal{I}^{\text{new}}$.

To prove assertion (1b), we note that assumption $\tilde{\kappa}^{\text{old}} \geq \tilde{\kappa}^{\text{new}}$ together with $\tilde{\kappa}^{\text{old}} < 0$ implies $|\tilde{\kappa}^{\text{old}}| \leq |\tilde{\kappa}^{\text{new}}|$, and therefore

$$1 + |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \leq 1 + |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|}$$

for all $a \in H(\beta)$. On the other hand, we have by assumption that

$$
\begin{aligned}
1 - |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} &\geq 1 - \max_{a \in L(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \\
&\geq 1 - \min_{a \in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \geq 1 - \max_{a \in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} > 0,
\end{aligned}
$$

for all $a \in L(\beta)$. Hence, we find that $w^{\text{old}}(a) \geq w^{\text{new}}(a) \geq 0$ for all $a \in [0, 1)$ as before, and so the same conclusion as in (1a) holds true.

Under the assumptions of assertion (2a), total precision of the initial wealth allocation is given by

$$
\mu^{\text{old}} = \int_{L(\beta)} \frac{\nu(a)^2}{1 + |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}} \, \mathrm{d}a + \int_{H(\beta)} \frac{\nu(a)^2}{1 - |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}} \, \mathrm{d}a,
$$

and total precision of the new wealth allocation is given by

$$
\mu^{\text{new}} = \int_{L(\beta)} \frac{\nu(a)^2}{1 + |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|}} \, \mathrm{d}a + \int_{H(\beta)} \frac{\nu(a)^2}{1 - |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|}} \, \mathrm{d}a.
$$

Since $\psi(a, rv_0^{\text{new}}(a)) \leq \psi(a, rv_0^{\text{old}}(a))$ for all $a \in L(\beta)$, $\tilde{\kappa}^{\text{old}} \leq \tilde{\kappa}^{\text{new}}$ implies

$$
1 + |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \leq 1 + |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}
$$

for all $a \in L(\beta)$. On the other hand, it holds that $1 - \max_{a \in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} > 0$ and

$$
\begin{aligned}
1 - \max_{a \in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} &\leq 1 - \min_{a \in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \\
&\leq 1 - \max_{a \in H(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \leq 1 - |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}
\end{aligned}
$$

for all $a \in H(\beta)$. Hence, we find that $w^{\text{new}}(a) \geq w^{\text{old}}(a) \geq 0$ for all $a \in [0, 1)$, and therefore $\mu^{\text{new}} \geq \mu^{\text{old}}$.

Finally, to prove assertion (2b) we note that by assumption

$$
\begin{aligned}
1 + |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} &\leq 1 + \max_{a \in L(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \\
&\leq 1 + \min_{a \in L(\beta)} |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \leq 1 + |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|}
\end{aligned}
$$

for all $a \in L(\beta)$, and thus $w_L^{\text{new}}(a) \geq w_L^{\text{old}}(a)$ for all $a \in L(\beta)$. On the other hand, since

$|\tilde{\kappa}^{\text{old}}| \leq |\tilde{\kappa}^{\text{new}}|$, and because $\psi(a, rv_0^{\text{new}}(a)) \geq \psi(a, rv_0^{\text{old}}(a))$ for all $a \in H(\beta)$, it holds that

$$1 - |\tilde{\kappa}^{\text{old}}| \frac{\psi(a, rv_0^{\text{old}}(a))}{|\beta(a)|} \geq 1 - |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} \geq 1 - \max_{a \in H(\beta)} |\tilde{\kappa}^{\text{new}}| \frac{\psi(a, rv_0^{\text{new}}(a))}{|\beta(a)|} > 0,$$

for all $a \in H(\beta)$. The strict inequality holds by assumption. Hence, we have $w_H^{\text{new}}(a) \geq w_H^{\text{old}}(a)$ for all $a \in H(\beta)$. We may again conclude that $w^{\text{new}}(a) \geq w^{\text{old}}(a) \geq 0$ for all $a \in [0, 1)$, and therefore $\mu^{\text{new}} \geq \mu^{\text{old}}$. ■

References

Adler, R. and J. Taylor (2007). *Random Fields and Geometry.* 1st ed. Springer Monographs in Mathematics. Springer New York, NY.

Avdis, E. and S. Glebkin (2025). *CHILE.* Preprint.

Banerjee, S., R. Kaniel, and I. Kremer (2009). "Price drift as an outcome of differences in higher-order beliefs". In: *The Review of Financial Studies* 22.9, pp. 3707–3734.

Banerjee, S. and I. Kremer (2010). "Disagreement and learning: Dynamic patterns of trade". In: *The Journal of Finance* 65.4, pp. 1269–1302.

Banerjee, S. (2011). "Learning from prices and the dispersion in beliefs". In: *The Review of Financial Studies* 24.9, pp. 3025–3068.

Breon-Drish, B. (2015). "On existence and uniqueness of equilibrium in a class of noisy rational expectations models". In: *The Review of Economic Studies* 82.3 (292), pp. 868–921.

Brown, D. P. and R. H. Jennings (1989). "On technical analysis". In: *The Review of Financial Studies* 2 (4), pp. 527–552.

Campbell, J. Y. and A. S. Kyle (1993). "Smart money, noise trading and stock price behaviour". In: *The Review of Economic Studies* 60.1, pp. 1–34.

Crandall, M. G., H. Ishii, and P.-L. Lions (1992). "User's Guide to Viscosity Solutions of Second Order Partial Differential Equations". In: *Bulletin of the American Mathematical Society* 27.1, pp. 1–67.

Fleming, W. H. and H. M. Soner (2006). *Controlled Markov Processes and Viscosity Solutions.* 2nd ed. New York: Springer.

Foster, F. D. and S. Viswanathan (1996). "Strategic trading when agents forecast the forecasts of others". In: *The Journal of Finance* 51 (4), pp. 1437–1478.

Grossman, S. J. (1977). "The existence of futures markets, noisy rational expectations and informational externalities". In: *The Review of Economic Studies* 44.3, pp. 431–449.

Grossman, S. J. and J. Stiglitz (1976). "Information and competitive price systems." In: *The American Economic Review* 66.2, pp. 246–253.

Grundy, B. D. and M. McNichols (1989). "Trade and the revelation of information through prices and direct disclosure". In: *The Review of Financial Studies* 2 (4), pp. 495–526.

Han, J. and A. S. Kyle (2018). "Speculative equilibrium with differences in higher-order beliefs". In: *Management Science* 64.9, pp. 4317–4332.

He, H. and J. Wang (1995). "Differential information and dynamic behavior of stock trading volume". In: *The Review of Financial Studies* 8 (4), pp. 919–972.

Hellwig, M. F. (1980). "On the aggregation of information in competitive markets". In: *Journal of Economic Theory* 22.3, pp. 477–498.

Kalman, R. and R. Bucy (1961). "New results in linear filtering and prediction theory". In: *ASME* 83 (1), pp. 95–108.

Karatzas, I. and S. Shreve (1998). *Methods of Mathematical Finance*. Springer New York, NY.

Kyle, A. S. (1989). "Informed speculation with imperfect competition". In: *The Review of Economic Studies* 56.3, pp. 317–355.

Kyle, A. S., A. A. Obizhaeva, and Y. Wang (2017). "Smooth trading with overconfidence and market power". In: *The Review of Economic Studies* 85.1, pp. 611–662.

Liptser, R. and A. Shiryaev (2001). *Statistics of Random Processes. I. General Theory*. 2nd ed. Springer.

Makarov, I. and O. Rytchkov (2012). "Forecasting the forecasts of others: Implications for asset pricing". In: *Journal of Economic Theory* 147 (3), pp. 941–966.

Milgrom, P. and N. Stokey (1982). "Information, trade and common knowledge". In: *Journal of Economic Theory* 26.1, pp. 17–27.

Revuz, D. and M. Yor (1999). *Continuous Martingales and Brownian Motion*. 3rd ed. Springer Berlin, Heidelberg.

Rockafellar, R. (1970). *Convex Analysis*. Princeton Mathematical Series 28. Princeton, NJ: Princeton University Press.

Tirole, J. (1982). "On the possibility of speculation under rational expectations". In: *Econometrica* 50.5, pp. 1163–1181.

Townsend, R. M. (1983). "Forecasting the forecasts of others". In: *Journal of Political Economy* 91 (4), pp. 546–588.

Walsh, J. (1986). "Martingales with a multidimensional parameter and stochastic integrals in the plane". In: *Lectures in Probability and Statistics*. Ed. by G. del Pino and R. Rebolledo. Springer Berlin Heidelberg, pp. 329–491.

Wang, J. (1993). "A model of intertemporal asset prices under asymmetric information". In: *The Review of Economic Studies* 60 (2), pp. 249–282.

Wang, J. (1994). "A Model of Competitive Stock Trading Volume". In: *Journal of Political Economy* 102.1, pp. 127–168.

Williams, D. (1991). *Probability with Martingales*. Cambridge: Cambridge University Press.